Chapter 16

16. RESEARCH AND DEVELOPMENT NECESSARY FOR DESIGN

At present sufficient background exists to allow preliminary design studies to begin whenever a decision to proceed is reached. However, a vigorous research program must also be started at once because many data have to be established and estimates verified or improved in order to avoid unnecessary sacrifices of performance. This research program may be divided into several branches of science:

Theory of Structural Enlargement - The engineering steps which lead from established prototype structures to greatly enlarged units are based on certain mechanical concepts of critical design requirements for various parts that perform definite functions. However, no part can be ideally designed for just one function alone; they overlap. In order to take full advantage of past experience it will be necessary to undertake a careful analysis and weight breakdown of sundry past and presently progressing high altitude or long range missile projects, in order to ascertain just exactly to what specifications their essential parts are designed and why. Such analyses of larger units throw light on the degree to which concessions in shape, in margins of safety, in allowances for technological and practical exigencies should be made.

Structural Materials - It seems certain that extensive investigation of the properties of materials will be required, especially the materials to be used for the tank walls, lids, and bottoms. The possibility of using fabric bags deserves consideration. Suitable scaling and insulating materials must be used which at the same time prevent leakage of the fuel and liquid oxygen.

Tanks - Since much depends upon the weight efficiency of the design of the large fuel and oxygen tank, a serious investigation into the merits of competing tank designs is indicated. Since practical manufacturing, insulating, plumbing and other technical considerations enter into the problem, research will have to go hand in hand with design towards evolving the most advantageous compromise between the various conflicting requirements. Special methods may have to be developed to test specimen structures under simulated acceleration loads.

Stage Separation - Means to separate smoothly the subsequent stage units when the booster unit fuel is exhausted, will have to be developed on the basis of extensive research into the various techniques previously tried and proposed, or yet to be evolved. These studies may include experimental work with reduced and full size dummy missiles prior to application to the full-fledged test rounds. Problems of accurate timing and of minimizing the pitch, yaw, and roll disturbances of the boosted unit will have to be solved. Inter-stage communication of automatic regulation and command signals and its harmless discontinuation upon stage separation will also require research, development and experimentation.

Erection, Assembly, Logistics - The design of the large stage parts of the vehicle will have to meet requirements dictated by considerations of erection and assembly procedures and of shipment of parts and subassemblies to test locations. An investigation into practical handling and operating procedures and into the logistics of the entire project must therefore be completed before the mother and grandmother can be completely designed. No radically new erection methods are believed necessary; those well developed in experimental aircraft production should be reasonably applicable.

Rocket Fuels - The reasons for choosing Liquid Oxygen and Alcohol as the propellants have been explained in Chapter 6. There are, however, several other propellant combinations which have a higher specific impulse. An extensive research program should be initiated to gain information on motor design criteria, storage and handling problems and the problem of logistics of such fuels. Several propellant combinations which warrant immediate consideration are -

(1) Liquid Hydrogen and Liquid Oxygen
(2) Hydrazine and Liquid Oxygen
(3) Methylamine and Liquid Oxygen
(4) Liquid Ammonia and Liquid Oxygen
(5) Hydroboron and Liquid Oxygen
(6) Nitromethanes (as a monofuel)

Any of these after an intensive research program may eventually prove superior to the Liquid Oxygen-Alcohol combination when all aspects and technical implications are understood and weighed.

Rocket Motors - Motor research will have to be directed towards improvements in fuel mixing, cooling, combustion chamber shape, nozzle wall materials, ignition, and pressure control. This research is expected to lead to reduction of weight and increase in reliability, performance and efficiency of motors. Rocket motors are usually tested in test pits equipped with elaborate laboratory instrumentation and safety devices. For the largest stage a new test pit surpassing those now existent will have to be established.

Rocket Accessories - While no fundamental research problems are foreseen to be involved in the scaling up of present turbines and pumps to the sizes necessary to feed the larger stages, new problems arise with the introduction of pump control for thrust throttling. The pump will be called upon to operate near peak efficiency from full speed down to about two-thirds speed. How best to accomplish this will have to be determined by research. No extensive research into plumbing is immediately required, provided existing standard fuels are used. However, if some new fuel is contemplated, effect on requirements for insulation, line sizes, valves, pumps, etc., must be investigated.

Test Stands - The size and cost of the vehicle would preclude the firing of test rounds in a free flight vehicle for the mere purpose of testing operational features. Therefore, each stage will have to be fired in a vertical test stand for proof testing. A test stand suitable to test run the mother and particularly the grandmother units will be considerably bigger than the German V-2 test stand. Its flame deflection and cooling requirements will be severe. The development of such a test stand with all its service equipment will be a sizable enterprise in itself and should unfold gradually from test experience gained with the lesser daughter stages on suitably adapted smaller operational test stands now available. Among things requiring proof testing and measuring, may be listed: the length of life of the jet vanes, the magnitude of malalignment of the jet, time lag of controls, magnitude of the thrust and of the control forces, functioning of all pumping machinery, etc.

Chapter 16

Problems in Gasdynamics - A vehicle of the type discussed in this report will have to travel through air varying in density from sea level values of .0024 slugs per cubic foot to those of the order of 10^{-13} at 200 miles altitude. The atmospheric conditions encountered thus vary from the standard sea-level values to those of an extremely rarefied gas. The physical phenomena, consequently, vary considerably, and so, of course, do the methods of evaluation. It is convenient to distinguish the regions encountered by the ratio of the characteristic length parameter of the body to the mean free path of the gas molecules. It is also convenient to introduce, in addition to some suitable dimension ℓ of the vehicle, another length, the boundary layer thickness δ which in terms of the velocity U and the kinematic viscosity ν is related to ℓ by

$$\delta^2 \approx \frac{\nu \ell}{U} \qquad (1)$$

The quantity ν is found from elementary kinetics of gases as

$$\nu \approx \Lambda \bar{c} , \qquad (2)$$

where $\bar{c}$ is the mean molecular velocity and Λ the mean free path. Following Tsien[1] we may now immediately indicate regimes of flow by comparing with Λ and ℓ . δ

(a) $\Lambda << \delta$

(b) $\Lambda \sim \delta$

(c) $\delta < \Lambda < \ell$

(d) $\Lambda >> \ell$

(1) H. S. Tsien: Symposium on High Speed Aerodynamics. Cal. Inst. of Tech., March 1946. (Unpublished)

Chapter 16

Region (a) is the realm of aerodynamics, in the usual sense. (b) is the region in which slip phenomena occur in the velocity distribution near a boundary. Temperature discontinuities also occur at the wall. Region (c) is the least understood case since in this region deviations from the Maxwellian velocity distribution play a large part. (d) is sometimes called the "Knudsen" region in which pure gas-kinetic methods apply.

Speaking in more aerodynamical terms, it is possible to distinguish the four regions in terms of Mach number, M, and Reynolds number, R, where

$$R = \frac{U l}{\nu}, \quad \text{and}$$

$$M = \frac{U}{a} \approx \frac{U}{c}$$

where a denotes the velocity of sound. Then from equations (1) and (2) it follows that

$$\frac{\delta}{l} \approx \frac{1}{\sqrt{R}} \approx \sqrt{\frac{\Lambda}{l} \times \frac{1}{M}},$$

and we therefore have

$$\frac{\Lambda}{\delta} \quad \frac{M}{\sqrt{R}} \qquad (3)$$

A given problem, therefore, can be characterized by the mean free path and boundary layer thickness, or by the Mach number and the Reynolds number. For example, orbital conditions at 100 miles altitude correspond to $M \sim 17$ and $R \sim 60$. Hence, $\Lambda / \delta \approx 2$ and thus the problem is in the slip region.

The research problems encountered in these various realms are briefly listed in the following paragraphs. First the case of very large mean free path is discussed. Then the problems of larger densities are taken up. The problems of air resistance and heat transfer in rarefied gases which are encountered in the present analysis differ from most problems of this kind previously studied. The former, such as those related to the oil drop experiment, etc., concerned problems in which the mean velocity was small compared with the molecular velocities. The problem here involves velocities which are large compared with the random molecular velocities. Hence the problem of momentum and energy transfer encountered here is similar to the interaction of a molecular beam of nearly uniform velocity and direction interacting with a solid surface.

Experiment and theory are both not too well advanced in considerations of this type. It seems highly desirable to initiate a research program. This work should start with a general review of what is already known, both experimentally and theoretically, and then proceed to fill the gaps in the knowledge. To mention a few assumptions which have not yet been completely substantiated, we have:

1. The reflection of molecules from a surface is generally assumed to be diffuse. The molecules are assumed to leave in random directions.
2. The departing molecules are assumed to have velocities related to the temperature of the wall by means of an empirical "accommodation" coefficient, A, which varies between zero and one. A = 0 corresponds to the case of elastic impact, A = 1 to the case where the temperature of the molecules of the wall is equal to the wall temperature.

Both of these assumptions affect directly the evaluation of drag and heat transfer. For the present case of high velocities a complication arises due to the fact that the molecules will not leave the wall, but will form a surface layer. This effect cannot yet be considered in the calculations and further research is needed.

Conditions at somewhat higher densities are still more complicated. In regimes where the mean free path is comparable with the boundary layer thickness but not large compared to the body dimensions, complications arise due to the fact that deviations from the steady state have to be considered. The necessary additional terms in the equations of motion due to the deviations from a steady state can be calculated from the work of Chapman and Enskog. Tsien has pointed out that the additional terms in the Navier-Stokes equations involve third order differential quotients and hence a new boundary condition has to be added. The exact form of this boundary condition is at present unknown. The formulation of such a condition will again involve knowledge of the interaction between the gas and the solid wall.

The gasdynamical range appears to be the best understood at present and considerable research here is already under way. New problems related to designs such as the present one are, for example, the non-stationary heat transfer problems which have to be considered for ascent and landing. Questions like the drag curve at high values of M, stability in passing through the sonic velocity, etc., are obviously of great importance. However, there is little difference in the gasdynamical range between problems arising in ordinary missile design and problems of the design of space vehicles. Of course, detailed design problems will differ.

It should be emphasized here that aerodynamical research related to a space vehicle will have to make use not only of extensive investigation in wind tunnels, both supersonic and hypersonic, but will also have to use many tools of modern experimental physics. The molecular beam methods, for example, will be one of the required tools. It is clear that in many cases the research has to be analytical and not experimental. In low speed aerodynamical research the method of model tests simulating actual flight conditions is extensively used. This type of test becomes exceedingly difficult and in fact impossible for vehicles travelling at great speeds. This is most easily seen in the following example. The ratio of stagnation temperature T_o to free-stream temperature T for a body travelling at a Mach number M is

$$\frac{T_o}{T} = 1 + \frac{\gamma - 1}{2} M^2, \quad \text{where}$$

$$\gamma = \frac{C_p}{C_v}$$

Hence for $M = 10$ and $\gamma = 1.4$, $T_o/T = 21$. To produce the Mach number 10 in a wind tunnel, an adiabatic expansion process is used and the temperature drops from the value T_o in the supply section to $T_o/21$ in the test section. In free flight, on the other hand, the free-air temperature T is given, and the temperature $T_o = 21$ T is produced by compression due to the fast moving vehicle. It is evidently very difficult indeed to match both M and T in the wind tunnel as compared with flight, since this would require a temperature in the supply section of the wind tunnel 21 times larger than the ambient temperature in flight. Research must become more analytical, and a close cooperation between experiment and theory and also between aerodynamics and physics is an absolute necessity.

Chapter 16

Aerology - Knowledge of the properties of the upper atmosphere above the stratosphere is expected to be greatly advanced during the present year as the V-2 firing program and the Hermes, Bumblebee, WAC and Nike projects progress. Arrangements will have to be made to cooperate with these programs towards securing some of the most important aerological data at an early stage. The factors of primary interest are: air temperature, atmospheric composition ionization, radiation and wind. As the satellite project itself develops, it will gradually furnish its own means of pushing exploration into the realm of higher altitude and speeds to determine the laws which govern the behavior of moving bodies there. Some of the incomplete missile aggregates, notably the mother - daughter - baby stage test rounds when fired on ballestic trajectories over hundreds or a few thousand miles range should furnish excellent opportunities of penetrating these virgin regions. Suitable instrumentation with which these and other test missiles can be equipped should be developed as the project gets under way.

Jet Control Rudders.- The jet rudders for control of this vehicle while under thrust presents a major problem in that they will be required to stay in the jet about four times longer than any that have been used to date. The natural Graphite vanes on the German V-2 lasted about 30 seconds before erosion rendered them useless for control. It therefore will be necessary to enter into a research program to determine what material or combination of materials will be able to withstand the high gas velocities of the hot jet sufficiently long. A passable solution may be in the almost unexplored powdered metals and sintered ceramics.

Another approach may be to have a number of vanes set at a given angle to the jet exhaust and as control is required the vanes feed into the jet.

As the vanes erode away the control would feed more and more of the vanes into the jet. The solution of the problem lies at the end of a research program.

Servo System - A considerable amount of research will have to be directed towards the development of a suitable servo motor system to actuate the jet vane. For instance, the relative merits of electric, pneumátic, hydraulic systems have to be investigated for each stage. Means to balance the vanes in order to keep their torque and power demands in bounds will have to be studied.

The control of the servo motors by means of an automatic program pilot system, its stability and freedom from undesirable hunting will come in for extensive research and development. It will require gyroscopic and accelerometric response elements which themselves will have to be specially developed even though their fundamental principles have already been successful in the V-2 missile. To these will be added a radio altimeter, the development of which will constitute a program beginning with research into the special functional requirement of such a device, when working at unprecedented altitudes and flight speeds and furnishing input signals to an automatic control regulator.

Guide Beam and Command System.- Inasmuch as it is anticipated that provision will have to be made to signal corrective guide commands to the missile to ensure its precise entry into the desired orbit, a good deal of research and experimentation with various competing guide beam and guide command systems must be instituted in order to evolve a satisfactory technique for the satellite. Antenna systems will have to be developed for the satellite missile in all its stages. Research will be directed towards determining the most suitable wave lengths for communication with the missile and towards developing suitable tubes for them, if none are available. Preliminary trials of proposed ground installations, as well as air borne radio altimeters, command receivers and transponders, should be arranged in conjunction with other missile firing projects in progress, and subsequently, with test missiles specially adapted for this purpose. During the later stages such tests can undoubtedly be combined with flight tests of the daughter stages of the satellite project itself. The command system eventually charged with bringing the missile back to earth will also constitute an object for research and development. The electric power supply for the missile borne electric equipment will also have to be developed to suit the unusual conditions prevailing on the orbit in an environment without air and gravity.

Simulator.- In order to ensure that the complex regulator loop system will work smoothly and without undesirable hunting it seems imperative that a so-called simulator be built and operated over numerous test runs. Such simulators have proved themselves invaluable in conjunction with many projects, viz Roc, Azon, Bumblebee, Hermes, Nike, V-2, FX, Henschel, etc. The development of an electrical simulator for the satellite missile can be undertaken in such a manner that it can be quickly adapted to represent many varieties of control systems with different response characteristics, lags, coordination, overrides etc. The representation will include the automatic control functions as well as the beam commands when superimposed, and it will be made to take care of the gradual changes of missile weight, moment of inertia, aerodynamic reactions, thrust, etc. during each simulated flight.

Attitude Control in Orbit. - Reaction flywheels or recoils which are being considered for control of the missile's attitude or orientation with respect to its flight path, will require research and development to ensure that they function as desired under the peculiar environment of the satellit and that their control is sufficiently forcible. Simulator techniques may also be helpful here.

Telemetry, Trajectory Survey and Communication. - Telemetry from guided missile to ground stations is still in its infancy. Great strides towards its realization are expected to be made as the presently active projects, notably the V-2 program, Hermes, Bumblebee, etc., progress. Clos coordination with these activities will be mandatory. Later on, special firing of some missiles for the benefit of the perfection of satellite telemetering systems may be indicated. The same is true of the evolution of trajectory survey by photo and kine-theodolite and radar tracking gear. Eventually the entire communication system, which will be required in the actual satellite operation, will have to be developed, tried out and practiced. The development of the instrumentation for the "payload" will require specialized research along the lines of the various branches of science to which the data to be measured belong, notably, meteorological instruments, cosmic ray and ionization measurements, temperature and pressure measurements, radiation measurements, spectroscopy, photography, television, etc.

Digest of Existing Literature. - The scientists of many nations have written more or less technical or speculative articles on many objects pertinent to the satellite project. German scientists and engineers, in particular, have produced numerous reports covering almost every phase of their research and development work in conjunction with their missile projects, several of which have been technically radical advancements of the art. These reports are physically available in this country. They are, however, scattered between several agencies (Army Air Forces, Army Ordnance, Navy BuAer, Navy Ordnance and various other institutes.) They are in the process of being indexed, screened, abstracted and microfilmed. Some few have been translated. Translations are being made by specialists scattered throughout the nation-wide galaxy of activities in the field of guided missiles. One of the urgent tasks of the research agency of the satellite project would be to scan this wealth of literature, gather copies of translations of significant reports now available or in progress, obtain microfilms of others and organize the translations of those that have an immediate bearing on the yet unsolved or uncertain problems of the satellite project. In some instances it may be possible to secure from the military authorities the assistance of German authors or specialists to expedite the jobs of translating or abstracting the material.

Interrogations - In many instances discrepancies or conflicts appear in the literature between data, computations, theories and objectives of some phase or other of the German activities. Ad hoc questioning of German personnel held available in this country has already done much to clarify some of the problems involved. Even though these people have been interrogated many times it seems that as the digest of the literature and the transcript progresses and broadens, new questions arise continuously. It is therefore believed advisable to comprise in the research activities of the satellite project some set-up whereby discussions of such new questions can be quickly arranged between the German personnel and the American specialists on the problems.

Coordination with other projects - As has already been mentioned

Chapter 16

under various headings before, part of the necessary research activity will comprise close coordination with other high altitude and long range missile projects now actively being pressed forward in this country, and - if possible - abroad.

Continuity - Some of the research problems are of a fundamental nature and have a bearing on the first moves of the design staff. These will have to be tackled immediately. Others are directed towards results needed at various later stages of the program. These can be scheduled consecutively. A certain amount of overlap and continuity of research and development will therefore become necessary. Actually research and development will have to be continued to the very end of the project, the actual firing and navigation of the satellite missile is more a research task than a production job.

Chapter 17

17. CONCLUSION

In the preceding chapters, we have critically examined the possibility of designing a man-made satellite. This examination has been made within the strict limits of practical engineering analysis. We have found that modern technology has advanced to a point where it now appears feasible to undertake the design of such a satellite.

The magnitude of the task of establishing the first satellite vehicle in its orbit is impressively large. However, our analysis has shown that as experience is gained in this new field, the reduction in the magnitude of the task will be equally impressive.

Appendix A

A. THE UPPER ATMOSPHERE

In evaluating the performance of a very high altitude vehicle, such as that described in this report, it becomes necessary to have values for the physical properties of the upper atmosphere at extremely high altitudes, which heretofore were of little interest to the aeronautical engineer. Conditions in these high altitude regions have received some attention, both theoretical and experimental, in the past 20 or 25 years by a relatively small number of investigators. However, the present knowledge of the physical state of the upper atmosphere is far from complete, and as will become apparent in the course of the discussion, at the high levels there is quite some differences of opinion as to what the conditions are; at still higher levels there are practically no data or opinions available at all. In short, the knowledge of the atmosphere becomes more and more uncertain and speculative with increasing altitude. The knowledge which is available concerning the upper atmosphere is based essentially on the results of observations of meteors, the spectrum and height of the aurora, the behavior of radio waves, the anamalous propagation of sound, and the ionization of the atmosphere by solar radiation.

In general, workers in the field appear to be in fair agreement as to the atmospheric properties from sea level up to 60 miles altitude. Above this altitude the knowledge and agreement is much less definite. It should be mentioned at this point that since all of

Appendix A

the values given in the literature are based on the metric system, temperature in °K, altitude in km., the values which will be quoted here will be in terms of the same system. For the convenience of the reader, tables 4 and 5 for converting values of temperature and altitude to the English system, °R and ft., are given at the end of the text. The values finally adopted to represent the atmosphere will be presented in the engineering system of units.

Since the pressure at any altitude depends on the vertical distribution of temperature, and since the density depends on both temperature and pressure, it is evident that, of any of the atmospheric properties, the temperature is the most fundamental and important one to be considered. The discussion which follows is undertaken with this point in mind.

For the purpose of discussion, the atmosphere is usually divided into three main regions. The atmosphere from sea level to about 10 km. is referred to as the troposphere and that from 10 km. to 20 km. as the stratosphere. The region above 20 km., extending outward to interplanetary space (or however far outward the atmosphere may be considered to extend) is referred to as the upper atmosphere. As pointed out by Penndorf[1], measurements of auroral heights indicate the presence of atmosphere up to heights of 1000-1200 km. The average

(1) Penndorf, R.: Die Zusammensetzung der Luft in der hohen Atmosphäre. Meteorologische Zeitschrift, vol. 55; p. 30, 1938.

Appendix A

conditions in the troposphere and stratosphere are well known and form the basis for the standard atmosphere used in aeronautics, as given by Diehl[2].

The atmosphere above about 80 km. is strongly ionized and hence this region of the upper atmosphere from 80 km. outward is known as the ionosphere. The ionosphere is of fundamental importance in radio-wave propagation since it is owing to the reflection of these waves by the ionosphere that long distance radio communication is possible. It seems to be established that the ionization, and therefore the conductivity of the ionosphere, is caused by the ultra-violet solar radiation. Whether the particles responsible for the conductivity are ions or electrons has not been definitely established, especially for the lower levels of the ionosphere. A survey of the facts and theories of the ionosphere has been given by Mimno[3].

The ionosphere itself is divided into three main regions or layers. According to Berkner[4] the lower of these regions known as

---

(2) Diehl, W.S.: Standard Atmosphere - Tables and Data. NACA Technical Report No. 218; 1925 and 1940.

(3) Mimno, H.R.: The Physics of The Ionosphere, Reviews of Modern Physics, vol. 9, No. 1; Jan., 1937.

(4) Berkner, L.V.: Physics of The Earth - VIII, Terrestrial Magnetism and Electricity, McGraw-Hill; p. 451, 1939.

Appendix A

the E-region is moderately ionized and is situated in the vicinity of the 100 km. level. The next higher layer, the F_1-region, is more strongly ionized and is situated at about 210 km. Still higher and still more strongly ionized is the F_2-layer at about 300 km. Most of the present knowledge of the upper atmosphere is based on the study of these three regions plus some figures which have been deduced from meteor studies, the propagation of sound, and the spectrum of the aurora.

Since the aim of this study is to arrive at working values for a so-called standard upper atmosphere, the various data which are available, either experimental or theoretical, will be presented and from them, what appear to be the most reasonable deductions will serve as a basis for the final values to be adopted for use in the study of the performance of the satellite vehicle.

Gutenberg[5] has recently given values for the atmosphere from sea level to 100 km. These are presented here in Table 1, which has been copied from the paper by Gutenberg.

---

(5) Gutenberg, B.: The Physical Properties, Pressure, Temperature and Composition of the Upper Atmosphere. Aeronautical Symposium at the California Institute of Technology; March 1946.

TABLE 1.

Typical Data for the Atmosphere in Winter

(a) Northern Germany, (b) Southern California
(From Gutenberg, Ref. 5)

Altitude km	Temperature °C (a)	Temperature °C (b)	Density gr/m³ (a)	Density gr/m³ (b)	Sound Velocity m/sec (a)	Sound Velocity m/sec (b)	Mean Free Path, cm	Prevailing Wind, Vel. m/sec	Prevailing Wind, Dir. from	Pressure, millibars (a)	Pressure, millibars (b)
0	3	10	1280	1250	334	338	6.1×10^{-6}	5±	W	1015	1015
4	-17	-5	825	802	322	329	9.5×10^{-6}	increasing		607	618
8	-45	-35	525	528	304	310	1.5×10^{-5}	to tropopause		344	361
12	-52	-65	294	326	299	291	2.3×10^{-5}	20±	W	187	195
16	-50	-80	157	180	300	279	4.2×10^{-5}	decreasing		101	99
20	-48	-70	85	84	301	286	9×10^{-5}	variable		55	49
24	-44	-55	46	41	304	297	2×10^{-4}	variable		30	26
28	-20	-20	23	19	320	320	4×10^{-4}	increasing		17	14
32	+20	+10	12	10	345	339	7×10^{-4}	30±	E	10.2	8.7
40	+65	+35	5	4	370	353	.002	100±	E	4.4	3.4
50	+80	+70	1.6	1.2	376	372	.006	100±	E	1.6	1.2
60	+80	+75	0.5?		372	375	.02	150±	E	0.6	0.4
80	-10?		.08?		326?		0.1	150±	E	.07?	.05?
100	150?		.006?		415?		1	changing to Westerly		.007?	

TABLE 1
APPENDIX A

Hulbert,(6) another investigator in problems of the upper atmosphere, adopts the temperature distribution shown in Table 2 to represent the atmosphere up to 220 km.

Table 2

Temperature of the Day Atmosphere
(According to Hulbert, Ref. 6)

Altitude, km.	0	10	20	30	40	60	80	100	200	220
Temperature, °K	287	220	225	230	240	260	320	360	360	360

Fig. 1 copied from a paper by Martyn and Pulley(7) also gives information, actual and inferential, concerning the average vertical temperature disbribution. Above 100 km. it will be noticed that the temperature distribution has been extrapolated up to 300 km. as shown by the broken lines.

Fig. 2, copied from the paper by Penndorf, loc.cit. an equally reliable investigator, gives a somewhat different curve for the probable vertical temperature distribution above 100 km. The fact that Goetz(8), another well known worker in the field, uses Penndorf's curve may perhaps lend added weight to the values shown in Fig. 2. It will be noticed that the composition of the atmosphere, according to Penndorf, is indicated by the curve on the left hand side of Fig. 2.

---

(6) Hulbert, E.O.: Physics of The Earth - VIII, Terrestrial Magnetisum and Electricity, McGraw Hill; p.493, 1939.
(7) Martyn, D.F. and Pulley, O.O.; The Temperatures and Constituents of the Upper Atmosphere. Proc. Roy.Soc., A154; p.482,1936
(8) Goetz, F.W.P.: Ergebnisse Der Kosmischen Physik-Band III - Physik Der Atmosphäre,Leipzig, Akad.Verlog.; p.314,1938.

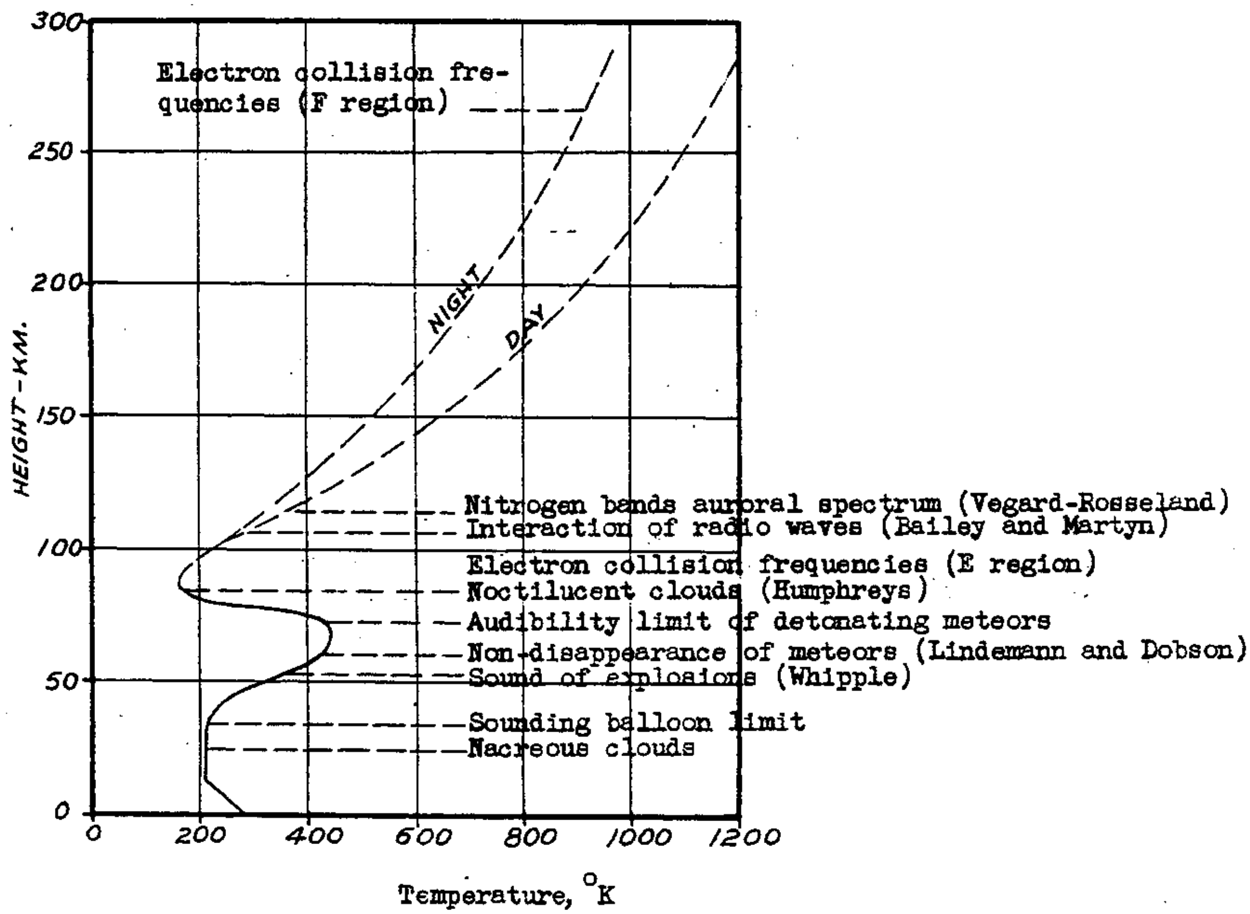


Fig. 1 Temperature Distribution in the Upper Atmosphere According to Martyn and Pulley (Ref. 7).

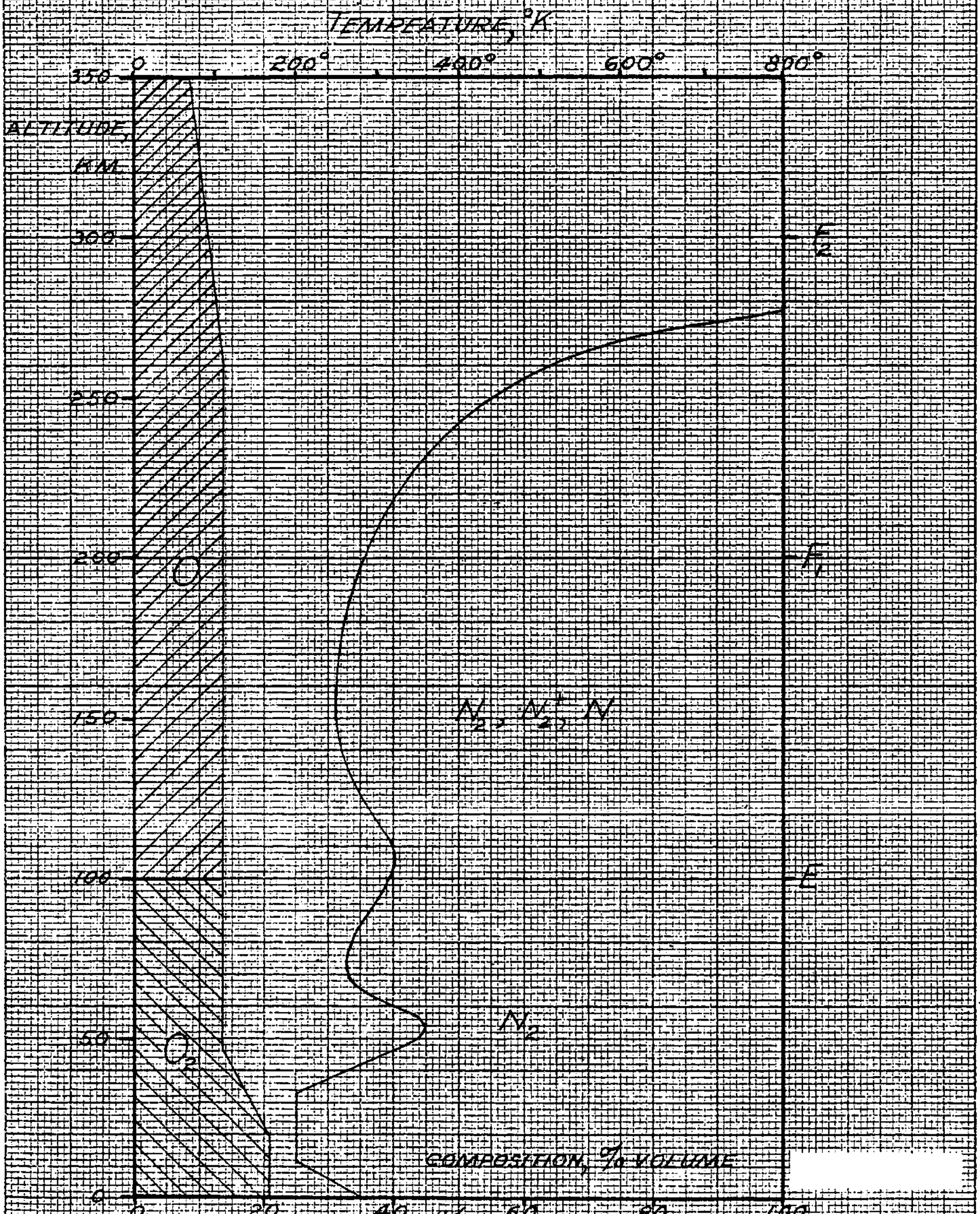


FIG. 2. VERTICAL DISTRIBUTION OF TEMPERATURE ACCORDING TO PENNDORF

Appendix A

The data presented thus far in Tables 1 and 2 and in Figs. 1 and 2, plus the standard atmosphere data of ref. 2 may safely be said to represent the present state of knowledge concerning the vertical distribution of the temperature of atmosphere up to altitudes of about 300 km. It is worth noting that none of the data extend beyond this level. Thus there is a relatively large region extending from 300 km. up to 1200 km. and higher in which, although it comprises only a small part of the atmospheric mass, the temperature conditions are more or less unknown.

That there is not even complete agreement in the region from 60 to 300 km. is immediately evident from a comparison of Table 1, Table 2, Fig. 1 and Fig. 2.

The data presented here, plus the results of other investigators which may be used to throw additional light on the problem, will now be used in order to arrive at the some final and definite values of the probable average vertical temperature distribution which may be adopted to represent a standard upper atmosphere.

From sea level up to 20 km. the atmospheric temperature has been determined by a great many direct measurements (sounding balloons) and the average conditions are well represented up to 65,000 ft. by the values given by Diehl (loc. cit.) for the NACA standard atmosphere. Owing to the universal acceptance and widespread use of the NACA standard atmosphere, it will be adopted as a representation of the atmosphere up to 65,000 ft.

Above the stratosphere, all of the data except that of Hulburt indicate a maximum in the temperature curve at 50-60 km. The existence of this maximum is fairly well established by the work of

Appendix A

F. J. W. Whipple[9] and Duckert[10] on the anomalous propagation of sound, the work of Gowan[11] and Dobson[12] on ozone, by the more recent investigation of F. L. Whipple[13] based on meteor observations, and also in a recent paper by Gowan[14]. The temperatures at 50-60 km. given by Gutenberg agree essentially with the most recent results of F. L. Whipple and Gowan and these values will be adopted. The temperature at this level given in Fig. 1 appear to be too high.

There is some evidence for a temperature minimum at about the 80 km. level but the value given in Fig. 1 appears much too low, see F. L. Whipple, ref. 13 and Martyn[14A], and we again adopt the more conservative value given by Gutenberg which agrees with that shown by Penndorf.

(9) F. J. W. Whipple: Quart. Jour. Roy. Met. Soc. Vol. 60; p. 80, 1934.
(10) Duckert, P.: Gerlands Beiträge Zur Geophysik, Supplement 1, p. 280, 1931
(11) Gowan, E. H.: Proc. Roy. Soc., Vol. A128, p. 531, 1930.
(12) Dobson, G. M. B.: Proc. Roy. Soc., Vol. A129; p. 411, 1930.
(13) Whipple, F. L.: Meteors and the Earth's Upper Atmosphere, Reviews of Modern Physics, Vol. 18, No. 4, p. 246, 1943.
(14) Gowan, E. H.: Note on Ozonosphere Temperatures. Aeronautical Symposium at the California Institute of Technology; March, 1946.

Appendix A

Near and above the 100 km. level another rise in temperature is required by the results of the studies of the auroral spectrum by Vegard[15] and Rosseland and Steensholt[16] which give a temperature about the same as that of Hulbert and which is about midway between the values given by Martyn and Pulley and that given by Gutenberg. We therefore adopt the value in Table 2 at the 100 km. level.

Above 100 km. the variation of temperature becomes much less definite although it is generally agreed that high temperatures must exist in the F_2-region at 250 or 300 km. Godfrey and Price[17] have shown, on the basis of radiation equilibruim, that the highest possible equilibrium day-time temperature in the F_2-region is about 3300°K, and that the actual equilibrium temperature may have any value between this figure and 230°K. However, these authors have shown that the existence of high temperatures of the order of 1000°K or more is a necessary consequence of the presence of appreciable oxygen at these levels.

Martyn and Pulley, loc. cit., also agree that the temperature of the F-region must be of the order of 1000°K, and Martyn (ref. 14a) more recently states that the high temperatures originally found to exist in the F_2-region as a result of electron collision frequency measurements is confirmed by Fuchs and by Appleton from measurements of the thickness of this region. The weight of evidence in favor of high temperatures in the F_2-region is very considerable.

---

(15) Vegard, L.: Geophysiske Pub. Oslo, No. 9, 1932
(16) Rosseland, S and Steensholt, G.: Univ. Obs. Oslo; Publ. No.7, 1933
(14a) Martyn, D.F.: The Upper Atmosphere. Quart. Jour. Roy. Met. Soc. vol. 65; p. 329, 1939
(17) Godfrey, G.H. and Price, W.L.: Proc. Roy Soc., vol. A163; p. 237, 1937

Appendix A

Although there is general agreement on the existence of high temperatures at around 300 km. there is certainly nothing which definitely fixes the shape of the temperature curve between 100 and 300km. On the one hand we have the extrapolated curve of Martyn and Pulley, Fig. 1, which indicates rapidly increasing temperature starting at 80km. and continuing to a maximum at around 300km. On the other hand there is the curve of Penndorf showing practically an isothermal condition from 100 to 200km. and then a very rapid increase in the F_2-region above 200km. As far as the computation of pressure and density is concerned, these two curves would lead to considerably different results. The use of Penndorf's curve would lead to low values of pressure and density at high altitudes, while the use of the curve of Martyn and Pulley would lead to relatively high values for these qualities.

Thus, although Figs. 1 and 2 are in agreement as to the value of the high temperature at 300km., they represent the two extremes by which this temperature is reached starting from 100km. As a reasonable compromise for the probable temperature variation from 100 to 300km., it has been decided to adopt a temperature variation in this region which is an average of these two extremes.

Above 300 km. there are hardly any data which would serve to extend the temperature curve to higher altitudes. We do know that the thickness of the total F-region (comprising both the F_1 and F_2-region) is estimated to be of the order of 200km. thick (Martyn and Pulley, loc. cit. p.469) so that above the 300 km. level one might expect the temperatures to decrease fairly rapidly. Above this level, the only figure which has come to the attention of the writer is a value of $70^{\circ}C$ at 1000km. which is quoted by Rosseland[18] and is due to Vegard.

(18) Rosseland, S.: Theoretical Astrophysics. Oxford University Press; p.237

Appendix A

This temperature is based on high aureral observations and was computed on the assumption of thermal equilibrium. That thermal equilibrium does not obtain under the conditions existing at these altitudes is generally recognized. However, for lack of anything better, the value of 70°C at 1000 km. will be adopted as giving some indication, at least, of the probable order of magnitude.

The final values adopted (as described above) to represent the vertical destribution of temperatures in the upper atmosphere are indicated by the temperature curve presented in Fig. 3.

The composition of the upper atmosphere will now be briefly considered. The composition of the air in the troposphere (sea level to 10-20km) as given by Paneth [19] is shown by Table 3.

From the Table it is seen that N_2 and O_2 account for 99 percent of the composition, by volume, of the lower atmosphere. According to Penndorf (ref. 1, p. 31) and to Chapman [21], the results of auroral spectroscopy indicate that even from 100km to 1000km., oxygen and nitrogen are still the main constituents of the atmosphere. Some twenty years ago, Chapman and Milne[20], it was thought that because of their low molecular weights, either hydrogen or helium must be the main constituent in the high atmosphere. However, it is now the

---

(19) Paneth, F.A.: Composition of the Upper Atmosphere - Direct Chemical Investigation. Quart. Journ. Roy. Met. Soc., vol. 65; pp. 304-310, 1939

(20) Chapman, S. & Milne, E.A.: The Composition Ionisation and Viscosity of the Atmosphere at Great Heights. Quar. Journ. Roy Met. Soc., Vol. 46; p. 379, 1929

(21) Chapman, S.: The Upper Atmosphere Quar. Journ. Roy Met. Soc., Vol. 65; p. 304, 1939.

Analysis PRELIMINARY DESIGN OF SATELLITE VEHICLE
Prepared by O. Grimminger
Date May 2, 1946

DOUGLAS AIRCRAFT COMPANY, INC.

Page 14A
Model #1033
Report No. SM-11827

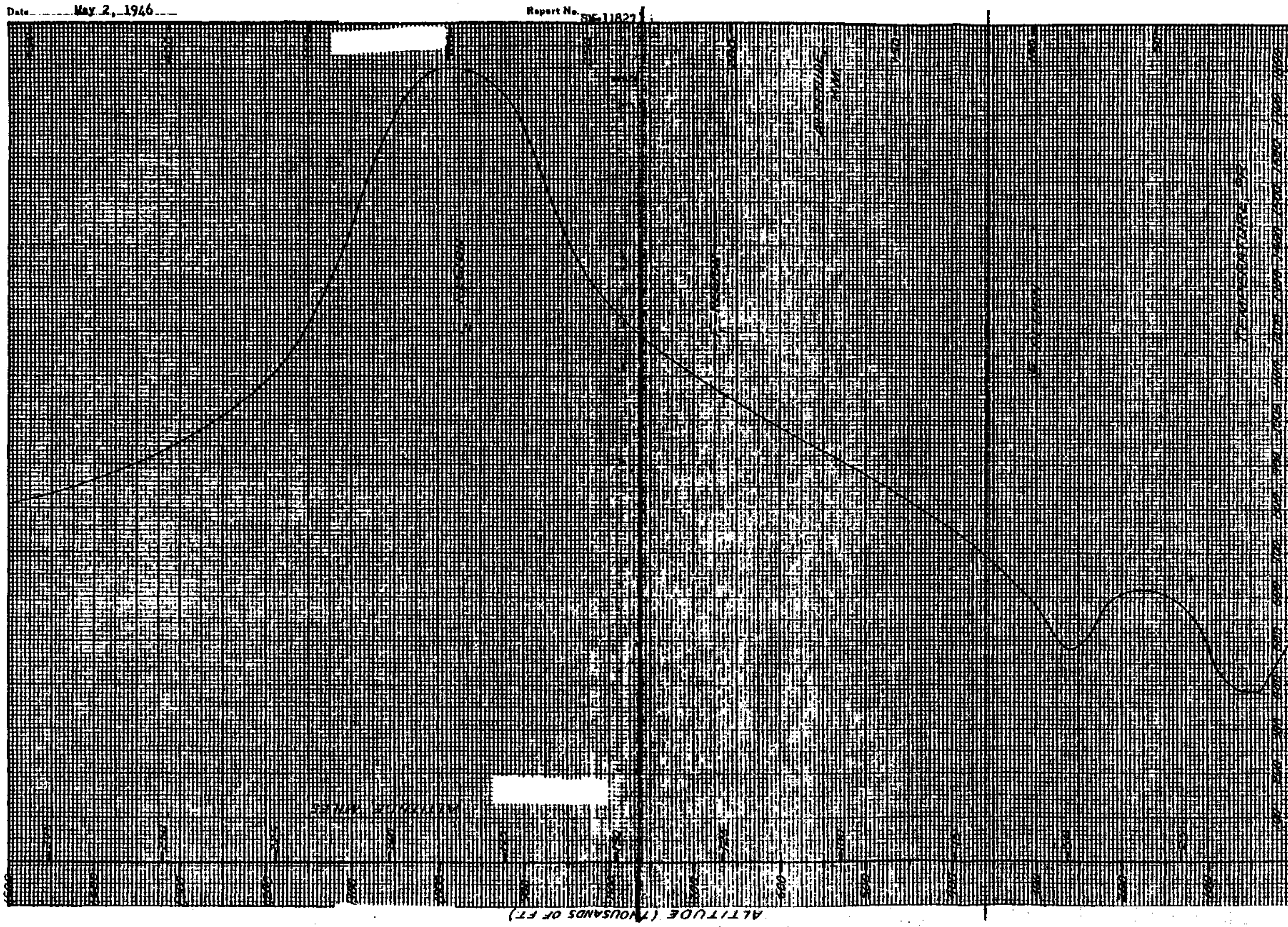


FIG.3. ADOPTED VERTICAL DISTRIBUTION OF TEMPERATURE TO REPRESENT CONDITIONS IN THE UPPER ATMOSPHERE

TWB

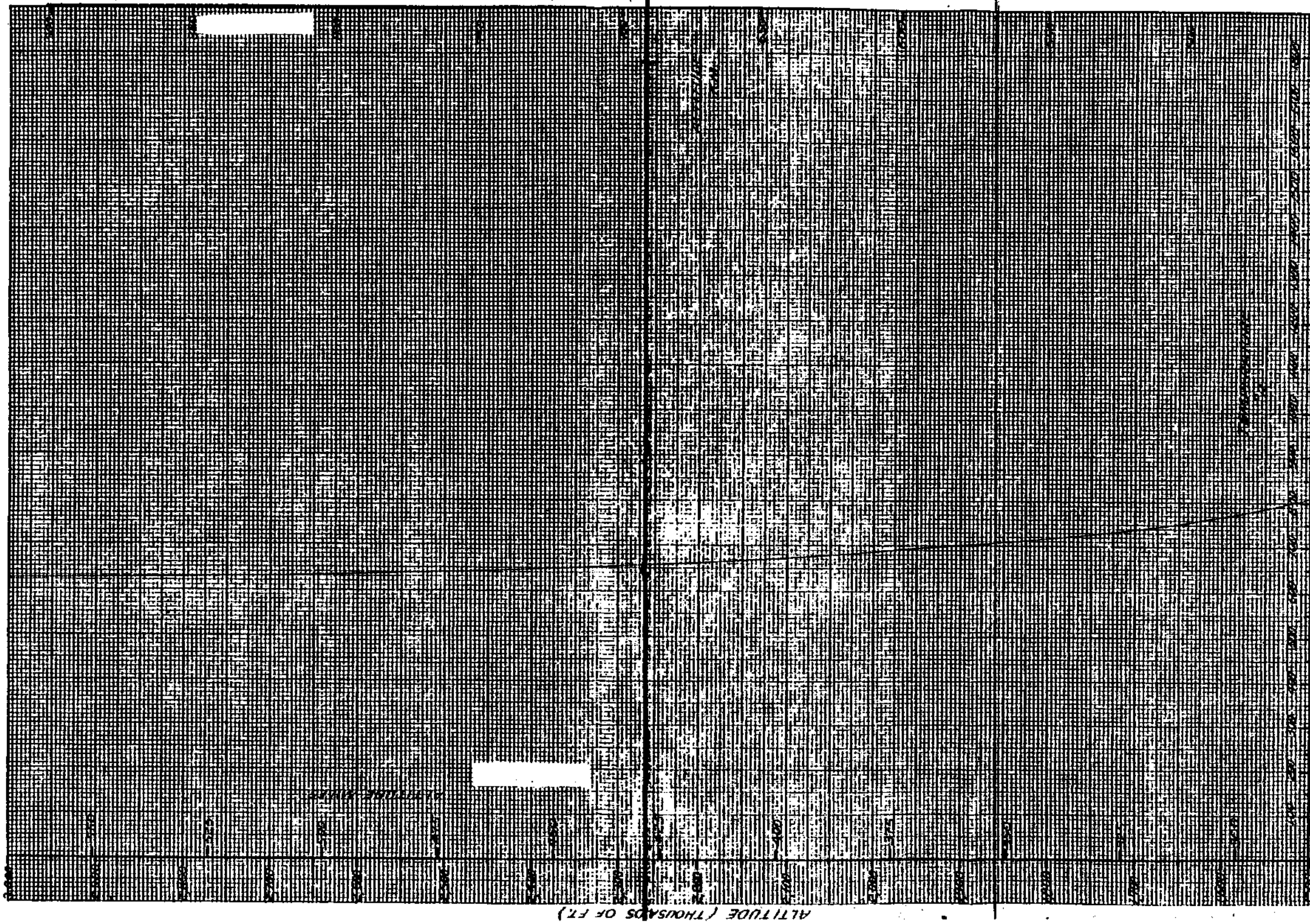


FIG. 3-(CONTINUED)

TWB

Appendix A

concensus of opinion that the upper atmosphere is a nitrogen-oxygen atmosphere, although the presence of hydrogen and helium has not been absolutely disproved, see Lindenmann, p. 331 of ref. (21). It will be assumed here that the upper atmosphere is a nitrogen-oxygen atmosphere.

TABLE 3

COMPOSITION OF TROPOSPHERIC AIR.AFTER PANETH.

Gas	Formula	Volume %	Molecular Wt. (0=16.000)	Density (Air=1)
Nitrogen	N_2	78.09	28.016	0.967
Oxygen	O_2	20.95	32.000	1.105
Argon	Ar	0.93	39.944	1.379
Carbon Dioxide	CO_2	0.03	44.00	1.529
Neon	Ne	$1.8.10^{-3}$	20.183	0.695
Helium	He	$5.24x10^{-4}$	4.002	0.138
Krypton	Kr	$1.10x10^{-4}$	83.7	2.868
Hydrogen	H_2	$5.10x10^{-5}$	2.016	0.0695
Xenon	X	$8.10x10^{-6}$	131.3	4.525

Chapman, ref. 21; and also Penndorf, Fig. 2, agree that the molecular oxygen O_2 must begin to undergo dissociation into atomic oxygen O beginning at 100-150km., and that the molecular nitrogen N_2 must undergo dissociation into atomic nitrogen at higher levels. In this report the following values will be adopted for the composition of the atmosphere.

Appendix A

Altitude Range	Composition	Molecular Weight of Mixture
0-150KM	21% O_2 and 78% N_2	28.7
150-500km.	12% O and 88% N_2	26.5
500 km. and higher	10% O and 90% N	14.2

Having adopted the temperature distribution shown in Fig. 3 to represent the probable average conditions in the upper atmosphere, the corresponding pressures are determined by use of the hypsometric equation (see Humphreys[22] or Diehl, ref.2), which is used here in the form

$$\log_{10} p_f = \log_{10} p_i - 0.000281\, m \left[1 - \frac{2h_f}{R}\right] \frac{h_f - h_i}{T_m}, \quad \frac{\text{lb}}{\text{sq.ft.}}, \qquad (1)$$

where h_i = altitude of lower level, ft.

h_f = altitude of upper level, ft.

p_i = pressure at the lower level, $\frac{\text{lb}}{\text{sq. ft.}}$

p_f = pressure at the upper level, $\frac{\text{lb}}{\text{sq. ft.}}$

R = radius of the earth = 20.89×10^6 ft.

m = molecular weight of the atmosphere.

T_m = the harmonic mean temperature in °R of the atmospheric layer of thickness $h_f - h_i$.

If the atmospheric layer from h_i to h_f is divided into n equal parts or intervals the harmonic mean temperature T_m is defined by

$$\frac{1}{T_m} = \frac{1}{n}\left[\frac{1}{T_1} + \frac{1}{T_2} + \text{-------} + \frac{1}{T_n}\right], \qquad (2)$$

(22) Humphreys, W.J.; Physics of the Air. McGraw Hill; pp.62-69,1929.

Appendix A

where T_m is the average temperature in the n^{th} interval. Thus to derive the vertical destribution of pressure, it is first necessary to have the temperature distribution curve, Fig. 3. Starting at some arbitrary level h_i where the pressure p_i is known, the pressure p_f at some higher level h_f is computed according to eq.(1). By dividing the atmosphere into a number of such layers the variation of the pressure with altitude is obtained. Since the NACA standard atmosphere was adopted here to represent the lower part of the atmosphere up to 65,000 ft., this altitude served as the starting point of the pressure calculations.

Knowing the pressure and temperature, the density is computed from the equation of state.

$$\rho = \frac{pm}{g_o R_u T}, \qquad (3)$$

where ρ = density, $\frac{\text{slugs}}{\text{ct.ft.}}$

p = pressure, $\frac{\text{lbs}}{\text{sq.ft.}}$

T = Temperature, ^{o}R

R_u = universal gas constant = 1545 $\frac{\text{lb-ft}}{\text{lb-mole } ^{o}F}$

g_o = standard value for the acceleration of gravity = 32.17 $\frac{\text{ft}}{\text{sec}^2}$ = (Constant)

m = molecular weight of the atmosphere.

Using this system of units, the equations for computing the density is written.

$$\frac{pm}{49600T} \qquad (4)$$

The adopted temperatures, from Fig. 3, and the corresponding pressures and densities, computed as described above, are tabulated in Table. 3.

TABLE 3

VALUES OF TEMPERATURE, PRESSURE, AND DENSITY OF THE ADOPTED UPPER ATMOSPHERE.

ALTITUDE (Thousands of feet)	ALTITUDE KM.	ALTITUDE Miles	Temperature, °R	Pressure, lbs/sq. ft.	Density, slugs/cu.ft.
0	0	0	518.4	2118	2.378×10^{-3}
35.3	10.76	6.64	392.4	490	7.27×10^{-4}
65	19.81	12.32	392.4	119.8	1.76×10^{-4}
100	30.48	18.95	500	26.01	3.06×10^{-5}
150	45.72	28.93	611	5.06	4.90×10^{-6}
200	60.96	37.90	628	1.17	1.11×10^{-6}
250	76.20	47.86	490	2.24×10^{-1}	2.67×10^{-7}
300	91.44	56.85	580	3.73×10^{-2}	3.75×10^{-8}
400	121.92	75.80	755	2.56×10^{-3}	1.97×10^{-9}
500	152.40	94.75	880	2.93×10^{-4}	1.78×10^{-10}
528	160.93	100.	906	1.81×10^{-4}	1.07×10^{-10}
600	182.88	113.70	984	5.28×10^{-5}	2.87×10^{-11}
700	213.36	132.65	1103	1.11×10^{-5}	4.38×10^{-12}
800	243.84	151.60	1264	2.89×10^{-6}	9.93×10^{-13}
900	274.32	170.55	1676	9.62×10^{-7}	2.49×10^{-13}
950	289.56	180.53	1788	6.16×10^{-7}	1.53×10^{-13}
1000	304.80	189.50	1798	4.00×10^{-7}	9.65×10^{-14}
1050	320.04	199.48	1700	2.58×10^{-7}	6.59×10^{-14}
1056	321.87	200.	1670	2.40×10^{-7}	6.24×10^{-14}
1100	335.28	208.45	1440	1.59×10^{-7}	4.79×10^{-14}
1200	365.76	227.40	1090	4.63×10^{-8}	1.84×10^{-14}
1300	396.74	246.35	950	1.06×10^{-8}	4.82×10^{-15}
1400	426.72	265.30	864	2.04×10^{-9}	1.03×10^{-15}
1500	457.20	284.25	810	3.55×10^{-10}	1.25×10^{-16}
1584	482.80	300.	774	1.58×10^{-10}	5.85×10^{-17}
1600	487.68	303.20	768	1.32×10^{-10}	4.93×10^{-17}
1700	518.16	322.15	760	4.75×10^{-11}	1.79×10^{-17}
1800	548.64	341.10	716	1.67×10^{-11}	6.66×10^{-18}
1900	579.12	360.05	700	5.75×10^{-12}	2.35×10^{-18}
2000	609.60	379.00	688	1.97×10^{-12}	8.20×10^{-19}
2100	640.08	397.95	674	6.69×10^{-13}	2.84×10^{-19}
2112	643.74	400.	671	5.80×10^{-13}	2.48×10^{-19}
2200	670.56	416.90	661	2.27×10^{-13}	9.80×10^{-20}
2300	701.04	435.85	656	7.66×10^{-14}	3.35×10^{-20}
2400	731.52	454.80	646	2.35×10^{-14}	1.04×10^{-20}
2500	762.00	473.75	640	7.93×10^{-15}	3.54×10^{-21}
2600	792.48	492.70	633	2.69×10^{-15}	1.22×10^{-21}
2640	804.61	500.	632	1.75×10^{-15}	7.90×10^{-22}
2700	822.96	511.65	630	9.21×10^{-16}	4.19×10^{-22}
2800	853.44	530.60	629	3.18×10^{-16}	1.44×10^{-22}
2900	883.92	549.55	627	1.11×10^{-16}	5.04×10^{-23}
3000	914.40	568.50	625	3.90×10^{-17}	1.79×10^{-23}
3100	944.88	587.45	624	1.39×10^{-17}	6.36×10^{-24}
3168	965.60	600.	623	7.00×10^{-18}	3.21×10^{-24}
3200	975.36	606.40	620	4.98×10^{-18}	2.30×10^{-24}
3300	1005.84	625.35	617	1.81×10^{-18}	8.37×10^{-25}

Table 3
Appendix A

TABLE 4
ALTITUDE CONVERSION

ALTITUDE			ALTITUDE			ALTITUDE	
KM.	THOUSANDS OF FEET	MILES	KM.	THOUSANDS OF FEET	MILES	MILES	THOUSANDS OF FEET
0	0	0				0	0
20	65.6	12.4	520	1706.	323.1	25	132
40	131.2	24.9	540	1772.	335.5	50	264
60	196.8	37.3	560	1837.	348.0	75	396
80	262.5	49.7	580	1903.	360.4	100	528
100	328.1	62.1	600	1968.	372.8	125	660
120	393.7	74.6	620	2034.	385.2	150	792
140	459.3	87.0	640	2100.	397.7	175	924
160	524.9	99.4	660	2165.	410.1	200	1056
180	590.6	111.8	680	2231.	422.5	225	1188
200	656.2	124.3	700	2297.	435.0	250	1320
220	721.8	136.7	720	2362.	447.4	275	1452
240	787.4	149.1	740	2428.	459.8	300	1584
260	853.0	161.6	760	2493.	472.2	325	1716
280	918.6	174.0	780	2559.	484.7	350	1848
300	984.2	186.4	800	2625.	497.1	375	1980
320	1050.	198.8	820	2690.	509.5	400	2112
340	1115.	211.3	840	2756.	298.2	425	2244
360	1181.	223.7	860	2822.	534.4	450	2376
380	1247.	236.1	880	2887.	546.8	475	2508
400	1312.	248.5	900	2953	559.2	500	2640
420	1378	261.0	920	3018.	571.7	525	2772
440	1444.	273.4	940	3084.	584.1	550	2904
460	1509	285.8	960	3150.	596.5	575	3036
480	1575	298.3	980	3215.	608.9	600	3168
500	1640	310.7	1000	3281.	621.4		

1 KM. = 3280.8 FT.

Table 4
Appendix A

TABLE 5
TEMPERATURE CONVERSION

°K	°C	°F	°R	°K	°C	°F	°R
0				500	227	440.6	900.6
20	-253	-433.4	26.6	520	247	476.6	936.6
40	-233	-387.4	72.6	540	267	512.6	972.6
60	-213	-351.4	108.6	560	287	548.6	1008.6
80	-193	-315.4	144.6	580	307	584.6	1044.6
100	-173	-279.4	180.6	600	327	620.6	1080.6
120	-153	-243.4	216.6	620	347	656.6	1116.6
140	-133	-207.4	252.6	640	367	692.6	1152.6
160	-113	-171.4	288.6	660	387	728.6	1188.6
180	- 93	-135.4	324.6	680	407	764.6	1224.6
200	- 73	- 99.4	360.6	700	427	800.6	1260.6
220	- 53.	- 63.4	396.6	720	447	836.6	1296.6
240	- 33.	- 27.4	432.6	740	467	872.6	1332.6
260	- 13.	8.6	468.6	760	487	908.6	1368.6
280	7.	44.6	504.6	780	507	944.6	1404.6
300	27	80.6	540.6	800	527	980.6	1440.6
320	47	116.6	576.6	820	547	1016.6	1476.6
340	67	152.6	612.6	840	567	1052.6	1512.6
360	87	188.6	648.6	860	587	1088.6	1548.6
380	107	224.6	684.6	880	607	1124.6	1584.6
400	127	260.6	720.6	900	627	1160.6	1620.6
420	147	296.6	756.6	920	647	1196.6	1656.6
440	167	332.6	792.6	940	667	1232.6	1692.6
460	187	368.6	828.6	960	687	1268.6	1728.6
480	207	404.6	864.6	980	707	1304.6	1764.6
				1000	727	1340.6	1800.6

$^\circ K = {}^\circ C + 273$

$^\circ R = {}^\circ F + 460$

Table 5
Appendix A

Appendix B

B. THE DETERMINATION OF THE DRAG COEFFICIENT.

In order to simplify the estimation of the drag coefficient for the purposes of this report, it was assumed that the vehicle was effectively conical in shape. Since drag, in almost all cases, is a second order effect, such an assumption is not out of order. The half cone angle, θ, was taken to be 0.3 radians. Other values of θ however, were eventually chosen for the more final designs.

At subsonic speeds, the total of pressure, friction, and base drag coefficients was taken to be 0.3, where the definition of C_D is

$$C_D = \frac{\text{Drag}}{(1/2)\rho V^2 A}$$

V is the velocity.

ρ is the mass density of air.

A is the frontal area.

This value was held constant for $0<M<0.8$.

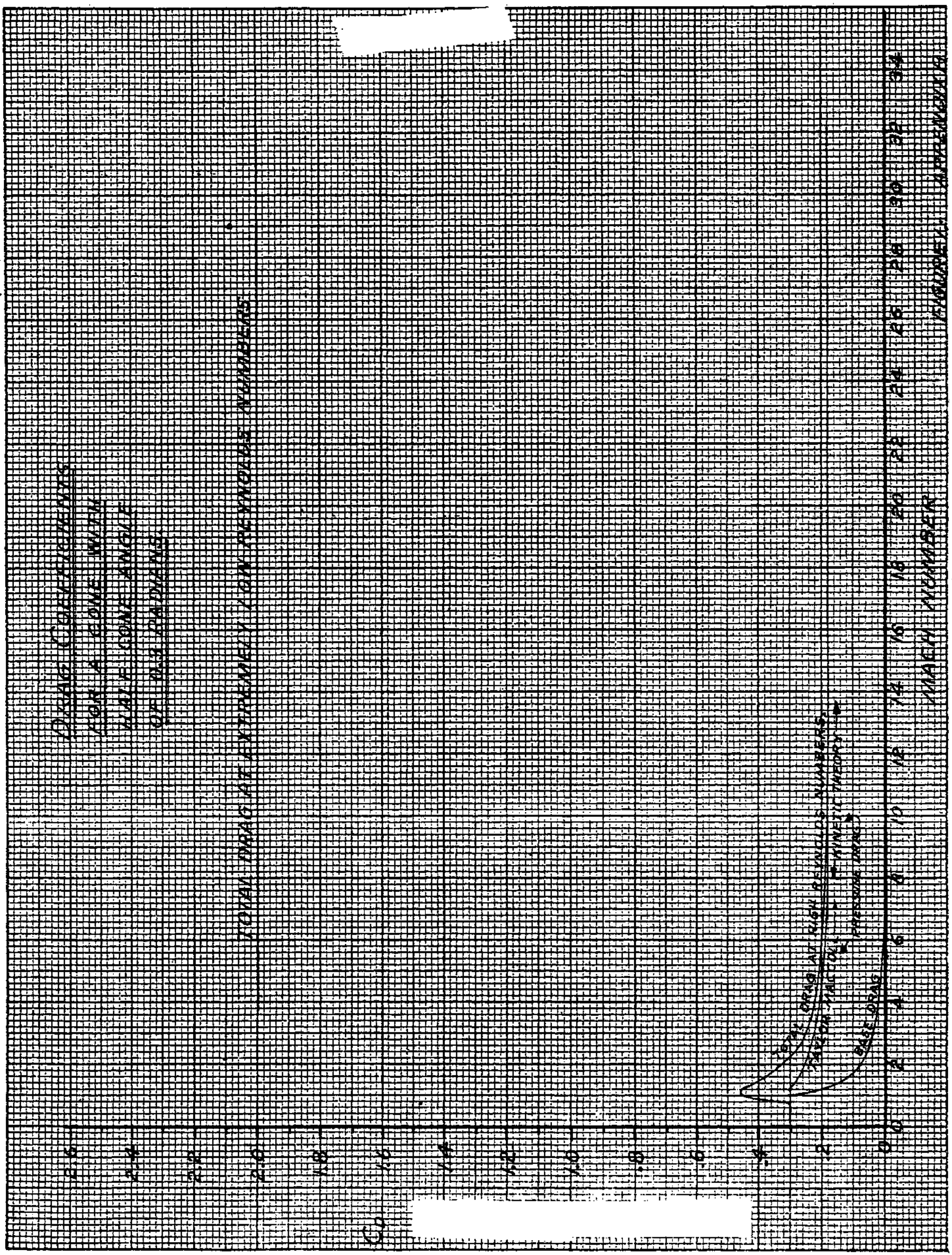
At low supersonic speeds, the well known work of Taylor and Maccöll is available, and gives values which are shown on the curve on fig. 1. Kinetic theory, under the assumption of inelastic impacts destroying the normal component of momentum, gives $2\theta^2$ for the hypersonic pressure drag coefficient. The supersonic base pressure coefficient, $(2/3)\frac{1}{\gamma M^2}$, was reduced to $\frac{.3}{M^2}$ because the base area of the rocket jet did not contribute drag. Skin friction, when based on frontal area, was sufficiently small to ignore (0.020) at high Reynolds numbers.

In the transonic region, a total drag coefficient of .45 was selected. This value is also shown on the accompanying graph, fig. 1.

Analysis DRAG OF CONE
Prepared by HAROLD LUSKIN
Date 5-6-46

DOUGLAS AIRCRAFT COMPANY, INC.

SANTA MONICA Plant



Appendix B

The values discussed above were used in the calculations for the ascent to the orbital height. The importance of the drag in those calculations gradually decreased, until at about 150,000 ft. above sea level, it was negligible compared to the thrust.

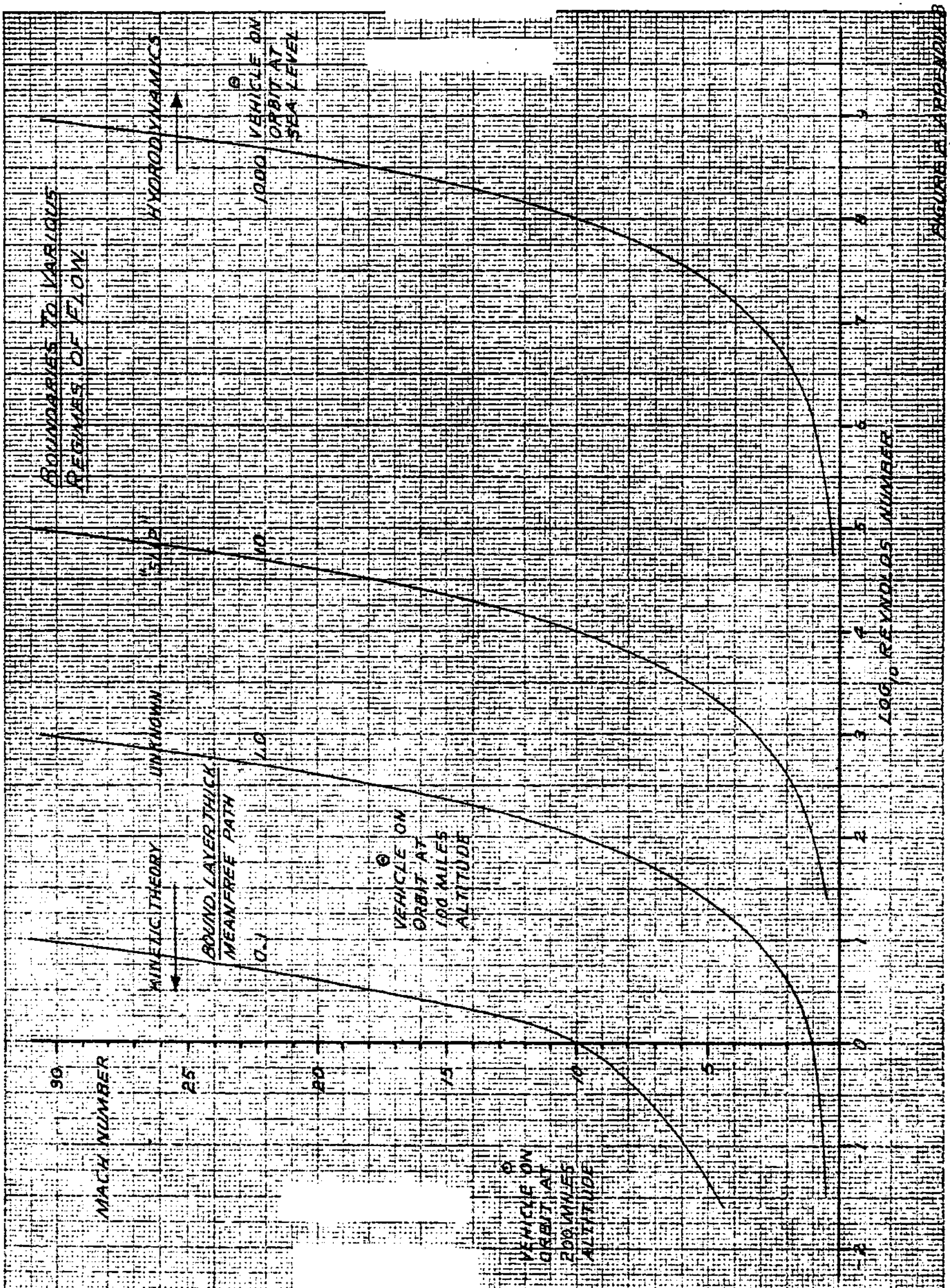
In cases where long periods of time are involved, especially where also the Reynolds numbers are low, it is necessary to consider carefully drags which would otherwise seem to be negligible. The detailed analyses of Sänger and others in Germany have given results for high altitude, high M conditions similar to those obtained by considering the destruction of all momentum in a cylinder whose cross-section is the same as that of the vehicle, and which approaches the vehicle at the vehicle's speed. Under such assumptions, it is found that $C_D = 2.0$.

As discussed elsewhere in this report, there are four regimes of flow which can conveniently be characterized by the ratio of mean free path to boundary layer thickness, or by the ratio of Mach number to square root of Reynolds number. The plot on fig. 2 shows these regimes and where in these various realms, the space vehicle flies. It will be noted that on fig. 1 no drag data are given for the slip or unknown regions. In these regions much research must be done.

Analysis DRAG
Prepared by CLAEYS
Date 4-4-46

DOUGLAS AIRCRAFT COMPANY, INC.

Model #1033
Report No. SM. 11827

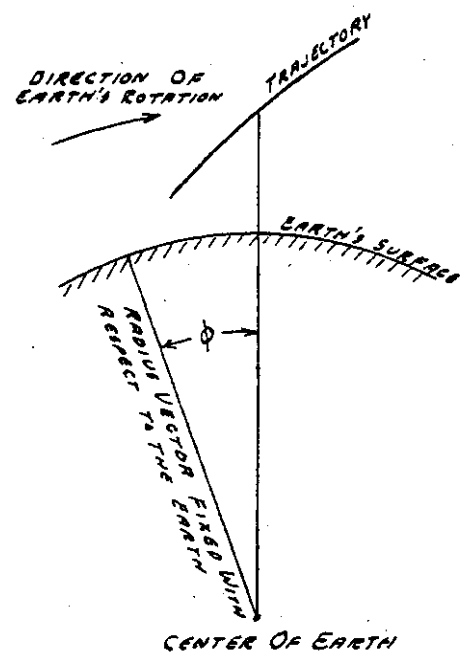


Appendix C

C. LAGRANGIAN EQUATIONS

<u>Development of the Equations of Motion of a Body Moving at Great Speeds Near the Surface of the Earth in the Plane of the Equator.</u> In this appendix, the equations of motion of a body will be developed in a form suitable for use in calculating the trajectory followed by the body as it is accelerated to the proper speed and direction for orbital motion. The analysis is confined to motion in the plane of the earth's equator because this is the only case considered in the calculations of the main text.

We shall take the following as variables characterizing the motion of the body: r, the radial distance to the body from the center of the earth; φ, the longitudinal angle of the body; and t, the time. We shall call m, the mass of the body; g, the acceleration of gravity; k, the gravitational constant; M, the earth's mass; R, the radius of the earth; and Ω, the angular velocity of the earth.



The kinetic energy of the body is

$$T = \frac{m}{2}\left[\dot{r}^2 + r^2(\varphi + \Omega)^2\right]$$

and the potential energy is

$$U = -\frac{kmM}{r}$$

Appendix C

If we form the La Grangian function L = T - U, then

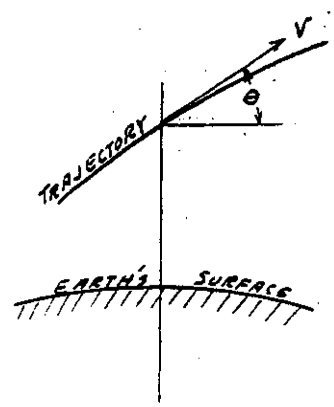
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = m\ddot{r} - mr\,(\dot{\varphi} + \Omega)^2 + \frac{kmM}{r^2} = F_r$$

and

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} = 2mr\,\dot{r}\,(\dot{\varphi} + \Omega) + mr^2\ddot{\varphi} = rF_\varphi$$

where F_r and F_φ are the radial and tangential components of all the externally applied forces.

We now rotate our coordinate system so that the vector resolutions are parallel and perpendicular to the trajectory. We designate by v the velocity of the body measured from a frame of reference fixed in the earth and by θ the complement of the angle between the radius vector and the tangent to the trajectory measured in coordinates fixed in the earth. The above equations become



$$m\ddot{r}\sin\theta - mr\,(\dot{\varphi}+\Omega)^2\sin\theta + 2\,m\dot{r}\,(\dot{\varphi}+\Omega)\cos\theta + mr\ddot{\varphi}\cos\theta + \frac{kmM\sin\theta}{r^2} = T\text{-}D\,,$$

$$m\ddot{r}\cos\theta - mr\,(\dot{\varphi}+\Omega)^2\cos\theta - 2\,m\dot{r}\,(\dot{\varphi}+\Omega)\sin\theta - mr\ddot{\varphi}\sin\theta + \frac{kmM\cos\theta}{r^2} = L\,,$$

where T-D is the thrust minus the drag, along the trajectory and L is the lift normal to the trajectory (positive when in the direction + π/2).

Appendix C

We now eliminate derivatives of r and φ by noting the following:

$$\dot{r} = v \sin \theta \qquad \dot{\varphi} = \frac{v \cos \theta}{r}$$

$$\ddot{r} = \dot{v} \sin \theta + v \cos \theta \,\dot{\theta} \qquad \ddot{\varphi} = \frac{\dot{v} \cos \theta}{r} - \frac{v \sin \theta \,\dot{\theta}}{r} - \frac{v^2 \sin \theta \cos \theta}{r^2}$$

These give, upon substitution in the equations of motion

$$m \dot{v} - m r \Omega^2 \sin \theta + \frac{kmM \sin \theta}{r^2} = T - D$$

$$m v \dot{\theta} - \frac{m v^2 \cos \theta}{r} - 2 m v \varphi - m r \varphi^2 \cos \theta + \frac{kmM \cos \theta}{r^2} = L$$

We can readily evaluate the gravitational constant because we know that when the body is standing still on the earth's surface, an external force mg is required to keep it in equilibrium. Putting $v = \dot{v} = \dot{\theta} = \theta = T - D = 0$ and $L = mg$ and $r = R$, we have

$$- m R \Omega^2 + \frac{k m M}{R^2} = m g \quad \text{or} \quad k M = g R^2 + R^3 \Omega^2,$$

Substituting this value of kM back into the equations of motion, we have

$$m \dot{v} - m r \Omega^2 \sin \theta + \frac{m(g R^2 + R^3 \Omega^2) \sin \theta}{r^2} = T - D$$

$$m v \dot{\theta} - \frac{m}{r} \cos \theta \,(v + r\Omega)^2 - 2m v \Omega (1 - \cos \theta) + \frac{m(gR^2 + R^3 \Omega^2) \cos \theta}{r^2} = L$$

It is of interest to investigate the significance of the various terms in these equations. In the first equation, which represents an equilibrium of forces in the direction of motion, $m \dot{v}$, T and D are the factors entering the familiar mass x acceleration = force. The term, $m r \Omega^2 \sin \theta$ is the component in the direction of motion of the centrifugal force caused by the earth's rotation. The term, $\frac{m(g R^2 + R^3 \Omega^2) \sin \theta}{r^2}$

Appendix C

is the corresponding component of the earth's attraction when account is taken of the fact that this attraction will impart an acceleration of 1g in the presence of the earth's rotation. In the second equation, which represents equilibrium of forces perpendicular to the direction of motion, in the plane of the equator, the term $m\,v\,\dot{\theta}$ is the centrifugal force as seen from local earth coordinates. The term $\frac{m \cos\theta}{r}(v + r\Omega)^2$ is the component of centrifugal force of the total rotation around the earth. The term, $2\,m\,v\,(1 - \cos\theta)$ can be interpreted as the apparent force that causes a body, ejected outward from the earth, to be left behind as the earth rotates under it. The term $\frac{m(g\,R^2 + R^3\Omega^2)}{r^2}\cos\theta$ is, of course, the component of the earth's attraction to normal to the direction of motion.

Using the above equations, we shall investigate the simple case of the free motion of the vehicle in a circular orbit at the earth's surface neglecting air resistance. For this case we put $L = \theta = \dot{\theta} = 0$; $R = r$ and the orbital velocity is determined by the relations

$$\frac{v^2}{R} + 2\,v\Omega + R\Omega^2 - g - R\Omega^2 = 0,$$

$$v = R\Omega \pm \sqrt{(R\Omega)^2 + g\,R}\ .$$

Using an equatorial radius of 3,963 miles and a value of g = 32.086, v = - 24,319 ft./sec. and 27,369 ft./sec. depending on whether the vehicle is moving with or against the earth's rotation.

It is seen that these values differ only slightly from the values of -24,285 ft./sec. and 27,335 ft./sec. given in Chapter III.

Appendix C

The values of Chapter III were computed from a simplified formula $v = R\Omega \pm \sqrt{gR}$ which neglects the effect of the earth's rotation on the apparent gravitational attraction of a stationary object. This difference in attraction is small, amounting to only about .6%.

Returning to the equation of motion to be used for the trajectory calculations, we shall put the altitude, $h = r - R$. For a trajectory 100 miles high, $\frac{h}{R} = 2\text{-}1/2\%$, a quantity whose square can be neglected compared to unity. Using this approximation, the equations take the form

$$\frac{dv}{dt} = 3H\Omega^2 \sin\theta - g\left(1 - 2\frac{h}{R}\right) \sin\theta + \frac{T - D}{m},$$

$$\frac{d\theta}{dt} = \frac{v}{R}\left(1 - \frac{h}{R}\right)\cos\theta + 2\Omega + 3\frac{h\Omega^2}{v}\cos\theta - \frac{g\left(1 - \frac{2h}{R}\right)}{v}\cos\theta + \frac{L}{m\,v}$$

If the terms in the above equation are examined for order of magnitude it is seen that $3h\Omega^2$ is always small compared to g. Consequently, we can maintain an accuracy of better than 1% using the following

$$\frac{dv}{dt} = -g\left(1 - \frac{2h}{R}\right) \sin\theta + \frac{T - D}{m},$$

$$\frac{d\theta}{dt} = \frac{v}{R}\left(1 - \frac{h}{R}\right)\cos\theta + 2\Omega - \frac{g\left(1 - \frac{2h}{R}\right)}{v}\cos\theta + \frac{L}{m\,v}$$

In the step by step calculations of trajectories, a preliminary calculation was usually made neglecting $\frac{h}{R}$ compared to unity and later the results were corrected for these small terms.

Appendix D

D. SAMPLE OF THE DETAILED TRAJECTORY CALCULATION

A = Maximum cross-sectional area of vehicle.

D = Drag,

g = Acceleration of gravity at earth's surface,

h = Altitude,

I = Specific impulse,

r = Distance from earth's center to vehicle,

R = Radius of earth,

t = Time

t_B = Burning time,

V = Velocity of vehicle,

V_E = Circumferential velocity of earth

W = Instantaneous mass of vehicle,

x = Distance projected on earth's surface,

ν = $\frac{\text{Original fuel weight per stage,}}{\text{Original total weight per stage}}$

α = Angle thrust makes with flight path (tilt),

θ = Angle of inclination of flight path to earth's horizontal.

Subscripts

ℓ = (lower) first three burning periods,

u = (upper) fourth burning period,

c = Coasting,

o = Initial condition,

1 = Beginning of an interval,

2 = End of an interval.

Appendix D

It is our purpose here to discuss the methods of calculating the trajectory. Neglecting $\frac{h}{R}$, the equations of motion are (see appendix C):

(1) $$\frac{dV}{dt} = \frac{g\,I\,\cos\alpha}{t_B\left(1-\gamma\frac{t}{t_B}\right)} - g\sin\theta - g\frac{D}{W}$$

(2) $$\frac{d\theta}{dt} = -g\cos\theta + \frac{V^2}{R}\cos\theta + \frac{2VV_E}{R} - \frac{g\,I\,\sin\theta}{t_B\left(1-\gamma\frac{t}{t_B}\right)},$$

(3) $$\frac{dh}{dt} = V\sin\theta,$$

(4) $$\frac{dx}{dt} = V\cos\theta.$$

When the term $\frac{g\,I\,\gamma}{t_B\left(1-\gamma\frac{t}{t_B}\right)}$ and $g\frac{D}{W}$ due to thrust and drag are absent, $\frac{h}{R}$ cannot be neglected. However, normally it can be neglected because $\frac{h}{R}$ occurs only in terms which are small compared to the rocket thrust terms.

We shall first review the general method of calculation, using as an example the four-stage alcohol-oxygen trajectory. Methods for the other c cases presented are similar. The vehicle travels vertically for half the time of the first burning period after which tilt is applied. The tilt remains constant until the end of the third burning period at which time coasting begins. In the last burning period a new angle of tilt is held constant. Because a knowledge of altitude is required in the calculations for the first and second periods and coasting, these computations were made as a group beginning at sea level and ending after coasting. Because the final velocity conditions are to a first approximation independent of altitude, we calculate the fourth period backward. From the first set of calculations, we obtain $\Delta\theta_\ell, \Delta V_\ell, \Delta h_\ell$ at the beginning of coasting for various values of γ and α_ℓ (plots shown in Figure D1, D2, D3). From the second set of calculations, plots (Fig. D4, D5, D6) of $\Delta\theta_u, \Delta V_u$, Δh_u versus γ and α_u are made. The coasting trajectories were computed from the equations for elliptic orbits which are discussed later.

To determine an actual trajectory, these plots and the coasting calculations are used to solve simultaneously the following equations:

$$\Delta V_\ell + \Delta V_c + \Delta V_u = \text{Orbital speed (depends on altitude)}$$

$$\Delta\theta_\ell + \Delta\theta_c + \Delta\theta_u = 90^\circ$$

$$\Delta h_\ell + \Delta h_c + \Delta h_u = \text{Desired altitude}$$

With four independent variables (γ, α_ℓ, α_u and Δh_c) and three restraining conditions, (orbital velocity, direction and altitude) we seek graphically the optimum trajectory for the γ of the proposed designs.

The actual step by step calculations in the burning periods differed somewhat between the burning periods. In the first burning period equation (2) had very little effect on equation (1) so that its effect on V as a function of t could be handled as a small perturbation of the results of (1) after its completion. In spite of this simplification, the calculations in the first burning period were the most difficult since the drag was large and the variation of I was considerable. If (1) is integrated for an interval in which I and $\frac{D}{W}$ could be considered constant, we have

Appendix D

$$V_2-V_1 = -g\,\bar{I}\cos\alpha\log\frac{1-\gamma\frac{t_2}{t_B}}{1-\gamma\frac{t_1}{t_B}} - g(t_2-t_1) - g\frac{\bar{D}}{\bar{W}}(t_2-t_1)$$

A second integration yields

$$h_2-h_1 = (V_1 + g\bar{I})(t_2-t_1) + g\bar{I}\frac{t_B}{\gamma}\left(1-\gamma\frac{t_2}{t_B}\right)\log\left(\frac{1-\gamma\frac{t_2}{t_B}}{1-\gamma\frac{t_1}{t_B}}\right) - g\left(1+\frac{\bar{D}}{\bar{W}}\right)\frac{(t_2-t_1)^2}{2}$$

The second equation is used to determine the altitude necessary for a knowledge of I and $\frac{D}{W} = \frac{C_D q A}{W_0(1-\gamma\frac{t}{t_B})}$ where I and the density in q depend only on altitude. The fact that (1) can be thus integrated enables us to to take much larger steps than a complete iteration process would require. Having established V as a function of t, we can use it in integrating (2) in a similar fashion:

$$\Delta\theta = \left(-\frac{g\,\overline{\cos\theta}}{\bar{V}} + \frac{\bar{V}\,\overline{\cos\theta}}{R} + \frac{2V_E}{R}\right)\Delta t + \frac{g\,\bar{I}}{\bar{V}}\log\frac{1-\gamma\frac{t_2}{t_B}}{1-\gamma\frac{t_1}{t_B}}$$

from this, we obtain θ as a function of t to correct the original velocity calculations.

In the second burning period I is constant and $\frac{D}{W}$ is negligible over most of the period. However, the variation in θ is now appreciable so that equations (1) and (2) must be iterated simultaneously.

The following remarks apply for all burning periods except the first. The partly integrated equations for a small interval are:

$$(1)'\quad \Delta V = -g\,\bar{I}\cos\alpha\log\left(\frac{1-\gamma\frac{t_2}{t_B}}{1-\gamma\frac{t_1}{t_B}}\right) - g\,\overline{\sin\theta}\,\Delta t$$

Appendix D

$$(2)' \; \Delta\theta = \left(-g\frac{\overline{\cos\theta}}{\overline{V}} + \frac{\overline{V}}{R} \quad \overline{\cos\theta} + 2\frac{V_E}{R}\right)\Delta t + g\,\bar{I}\,\sin\alpha \log \frac{1-\nu\frac{t_2}{t_B}}{1-\nu\frac{t_1}{t_B}}$$

Starting with (1)' we calculate ΔV to use in (2)' for $\overline{V}$ from which we get $\Delta\theta$ to use in (1)' for $\overline{\sin\theta}$ to get a better value of V. Altitudes need not be calculated until V and θ as functions of t are established and then they and the x's may be obtained by planimeter from equations (3) and (4). Sample calculations from each burning period are presented.

For coasting (rocket thrust absent) $\frac{h}{R}$ is not negligible so (1) and (2) become

$$\frac{dV}{dt} = -g\left(\frac{R}{r}\right)^2 \sin\theta,$$

$$V\frac{d\theta}{dt} = -g\left(\frac{R}{r}\right)^2 \cos\theta + \frac{V^2}{r}\cos\theta + 2\frac{VV_E}{R}$$

If the $\frac{V_E}{R}$ term is neglected these equations integrate into

$$V_2^2 - V_1^2 = -2gR\left(\frac{R}{r_1} - \frac{R}{r_2}\right)$$

and $\dfrac{\cos\theta_2}{\cos\theta_1} = \dfrac{r_1 V_1}{r_2 V_2}$, which are the equations of motion of a body in an elliptic orbit. When account is taken of the effect of the earth's motion the second equation is modified for sufficient accuracy into

$$\frac{\cos\theta_2}{\cos\theta_1} = \frac{r_1}{r_2}\frac{V_1}{V_2}e^{-\frac{2V_E}{V\cos\theta}\frac{\Delta r}{R}}$$

These equations are in the form used in the calculations.

ORM 25 6P

Analysis PRELIMINARY DESIGN OF SATELLITE VEHICLE

Prepared by W. H. Wampler

Date May 2, 1946

DOUGLAS AIRCRAFT COMPANY, INC.

Page 6D

Model #1033

Report No. SM11827

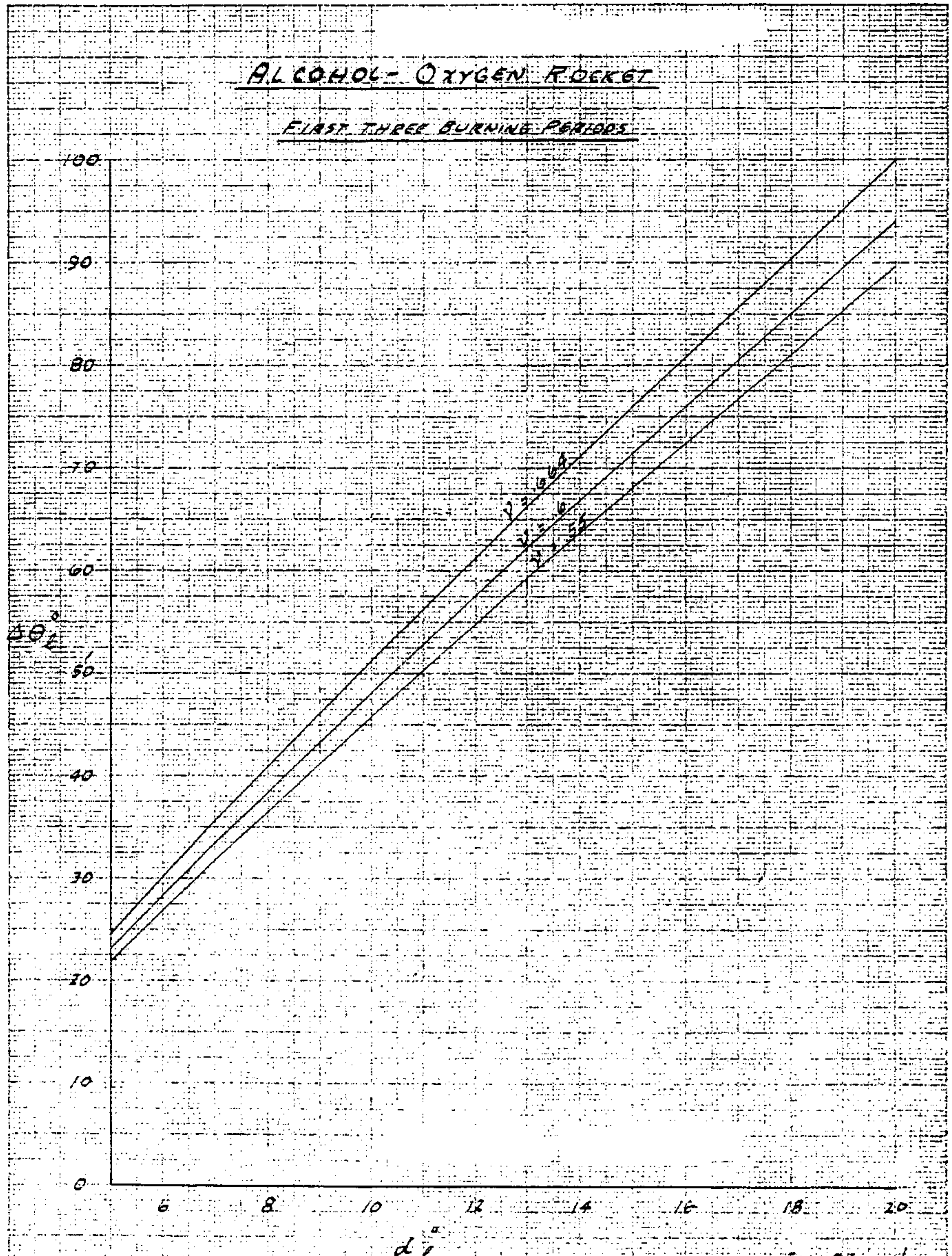


FIGURE 1
APPENDIX D

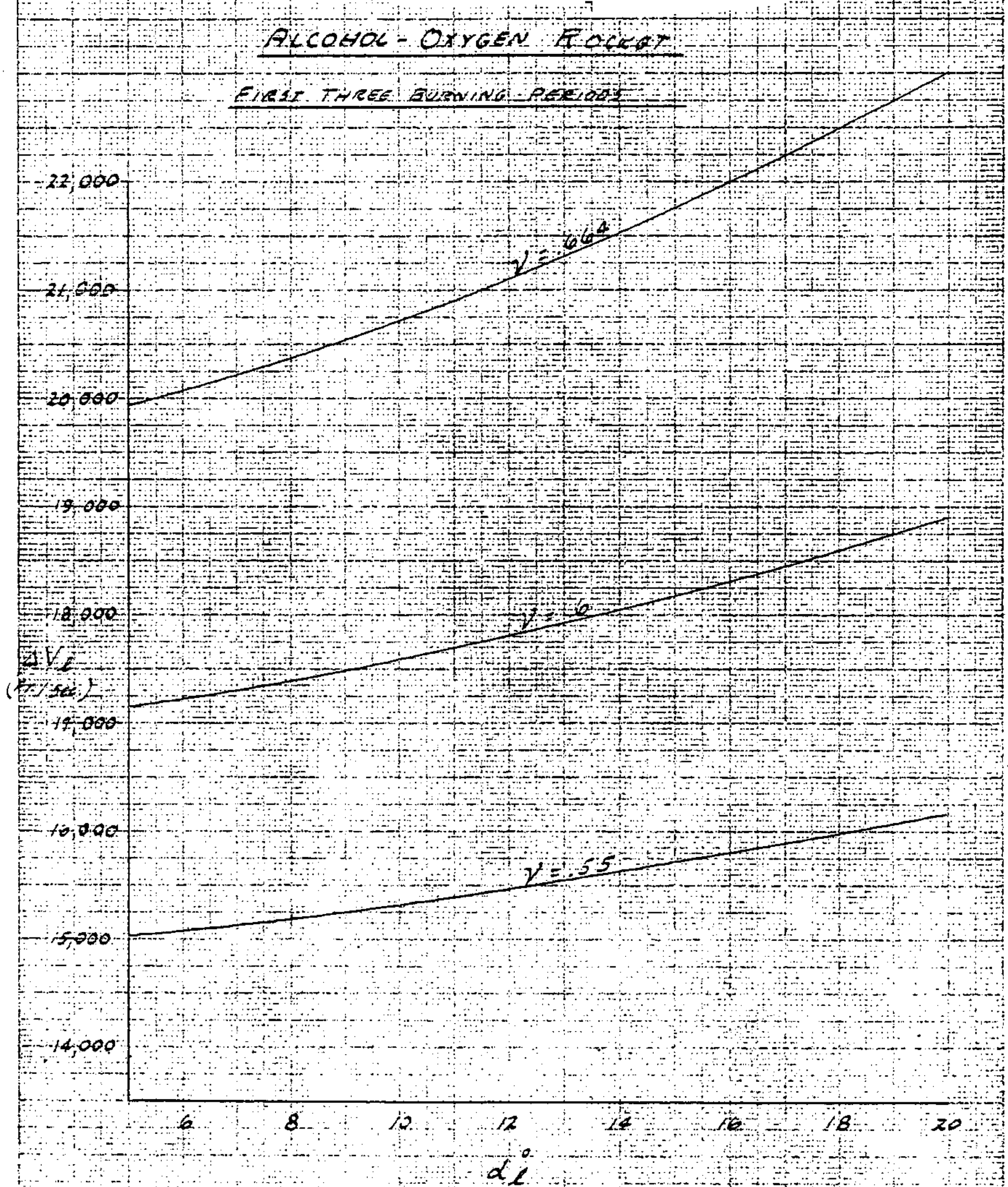


FIGURE 2
APPENDIX D

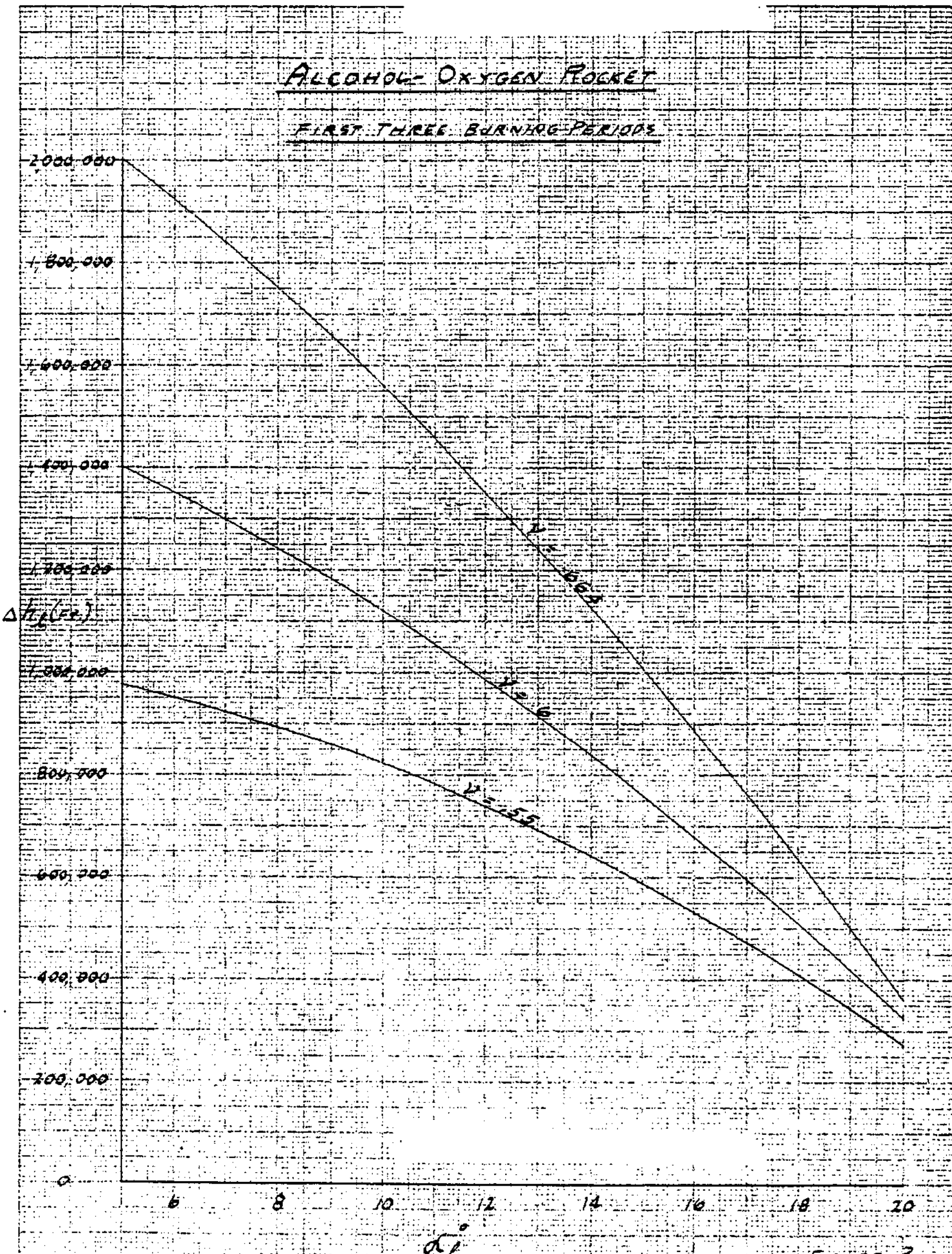

ALCOHOL-OXYGEN ROCKET
FIRST THREE BURNING PERIODS
2,000,000
1,800,000
1,600,000
1,400,000
1,200,000
1,000,000
800,000
600,000
400,000
200,000
0
Δh_ℓ (Ft.)
ν = 6.64
ν = 6
ν = 5.5
6
8
10
12
14
16
18
20
α_ℓ°

FIGURE 3

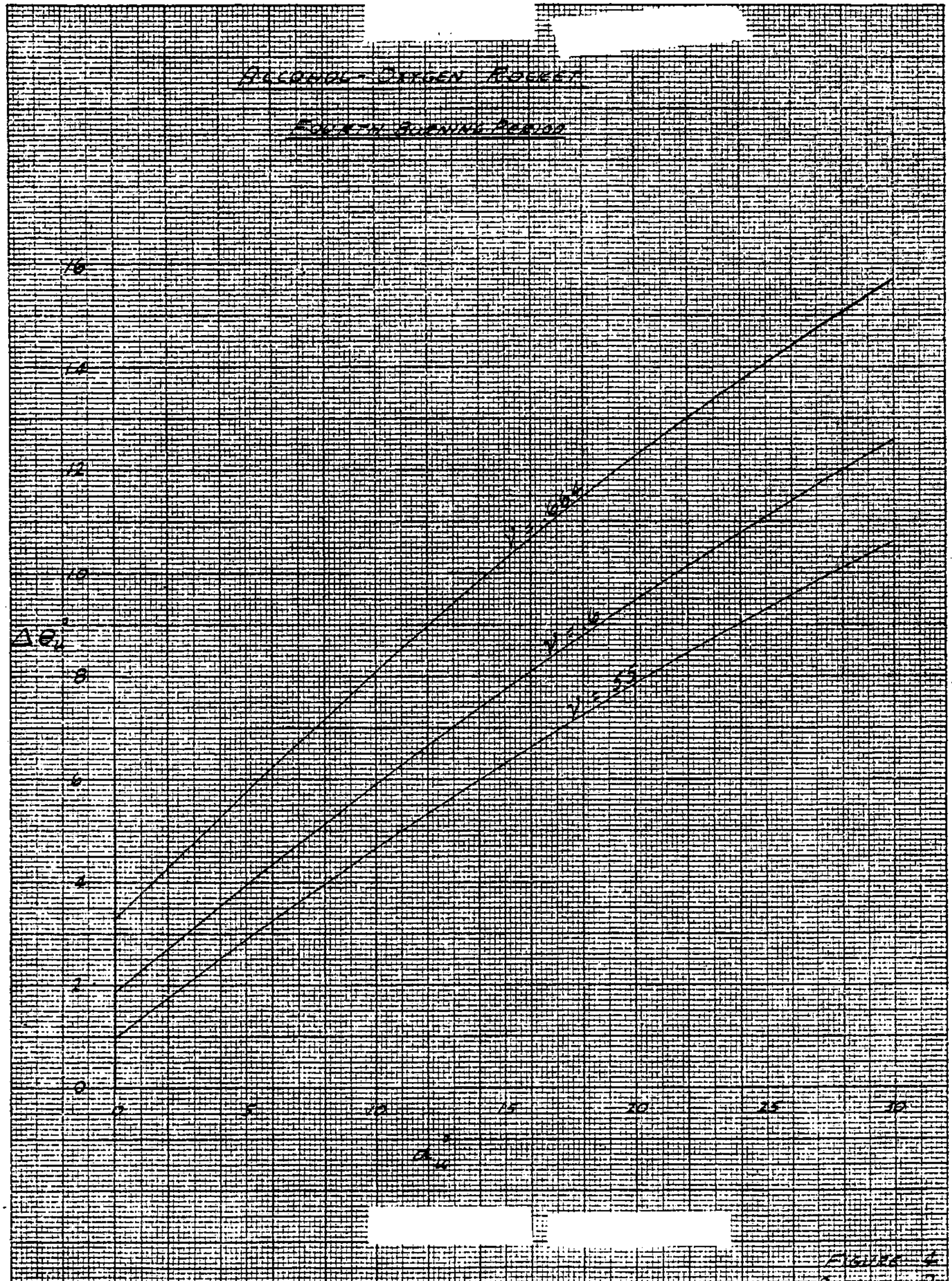


FIGURE 4
APPENDIX D

FORM 25 BP

Analysis PRELIMINARY DESIGN OF SATELLITE VEHICLE

Prepared by W. H. Wampler

Date May 2, 1946

DOUGLAS AIRCRAFT COMPANY, INC.

Model #1033

Report No. SM11827

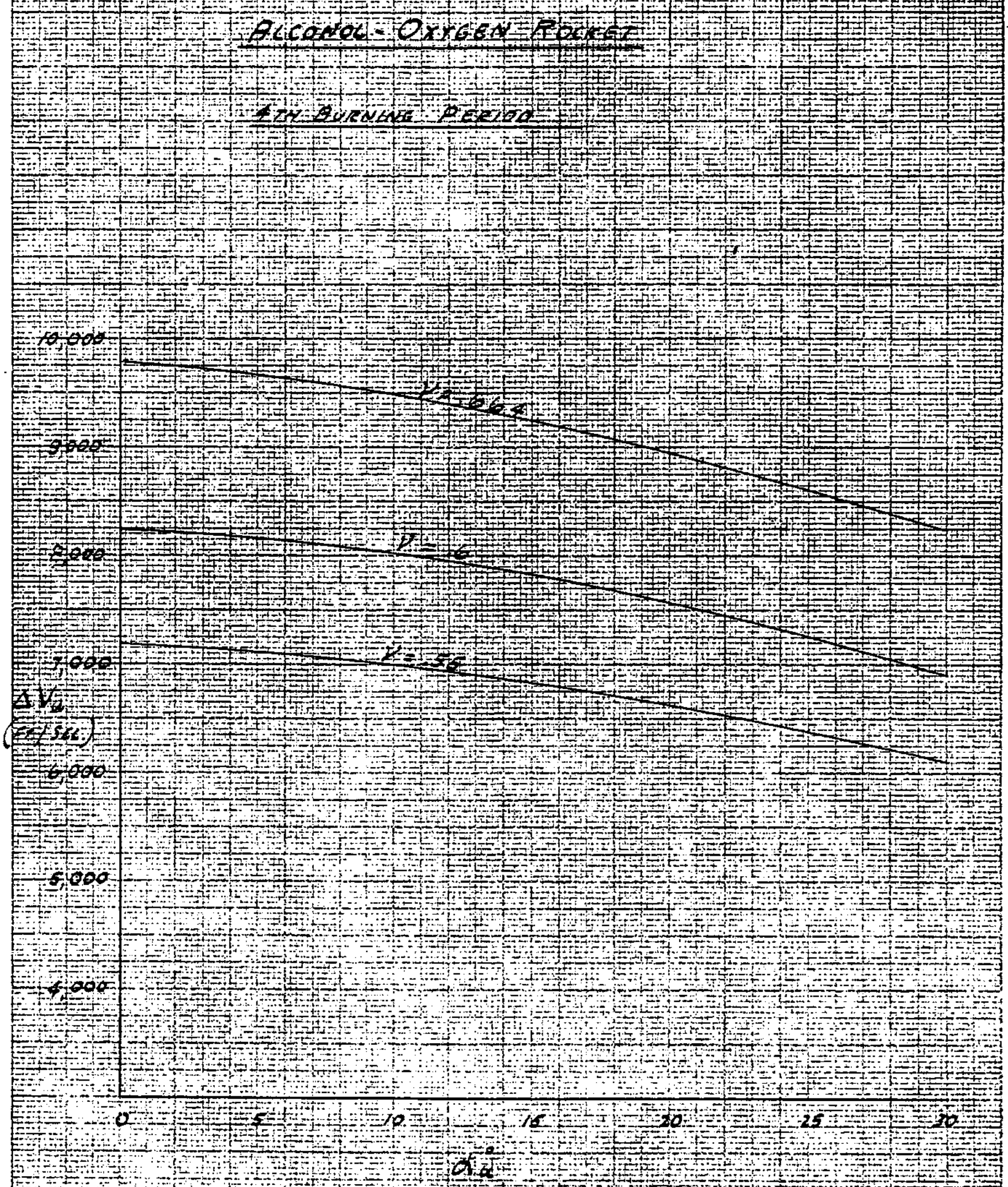


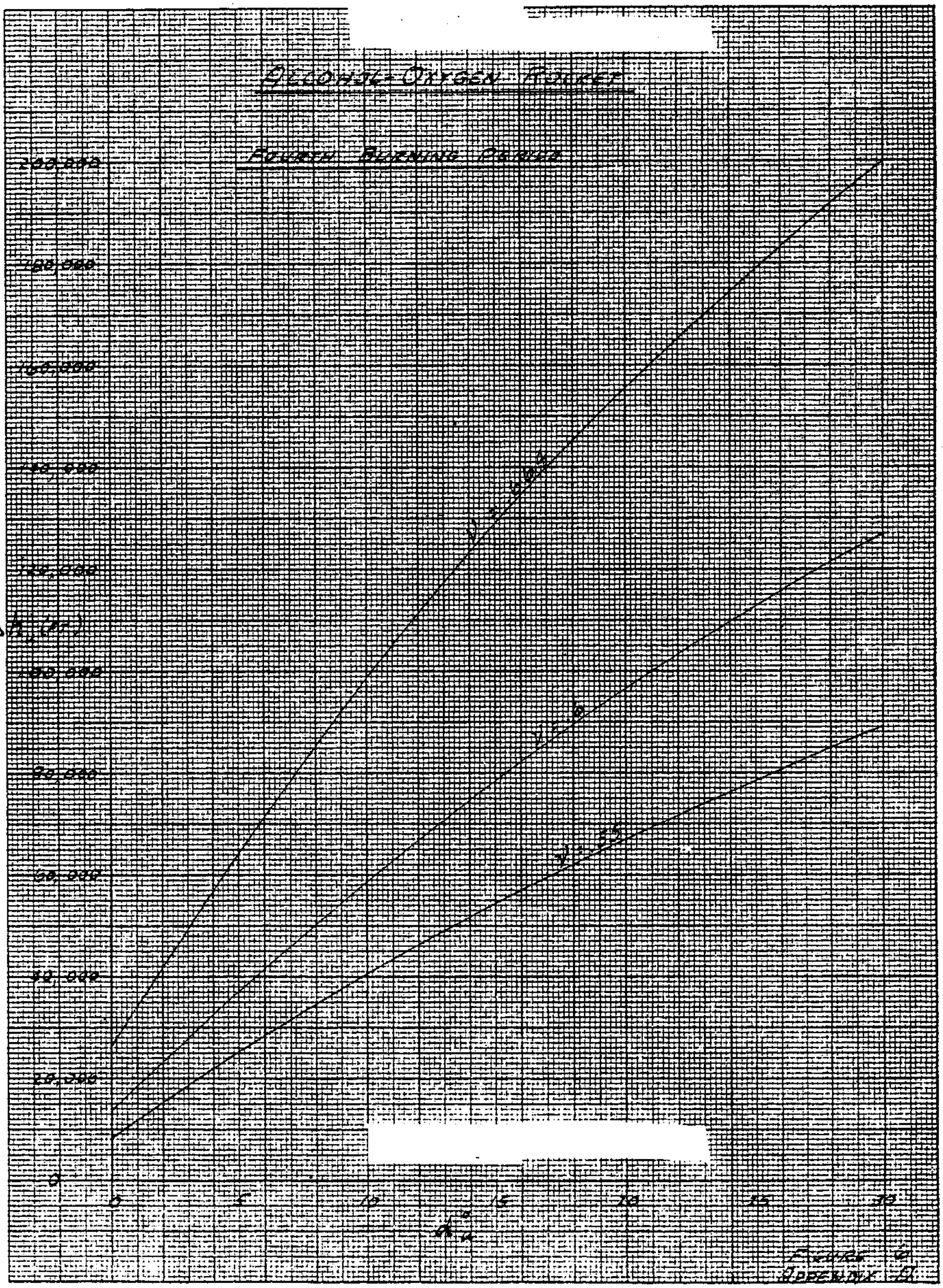
FIGURE 5

APPENDIX D.

Analysis PRELIMINARY DESIGN OF SATELLITE VEHICLE
Prepared by W. H. Wampler
Date May 2, 1946

DOUGLAS AIRCRAFT COMPANY, INC.

Page 11D
Model #1033
Report No. SM11827



FORM 23 8P

HYDROGEN - OXYGEN ROCKET

FIRST BURNING PERIOD

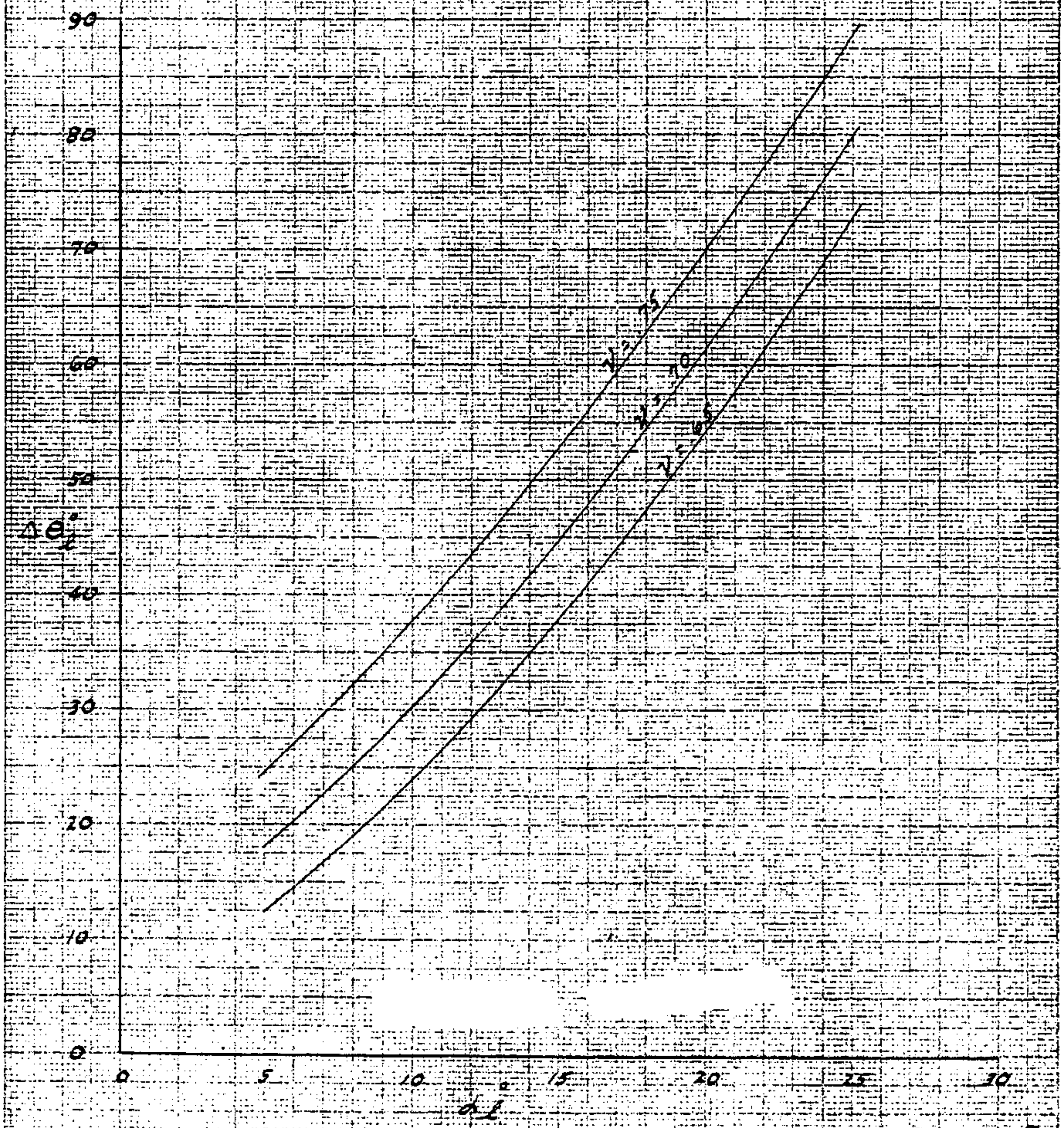


FIGURE 7
APPENDIX D

HYDROGEN – OXYGEN ROCKET

FIRST BURNING PERIOD

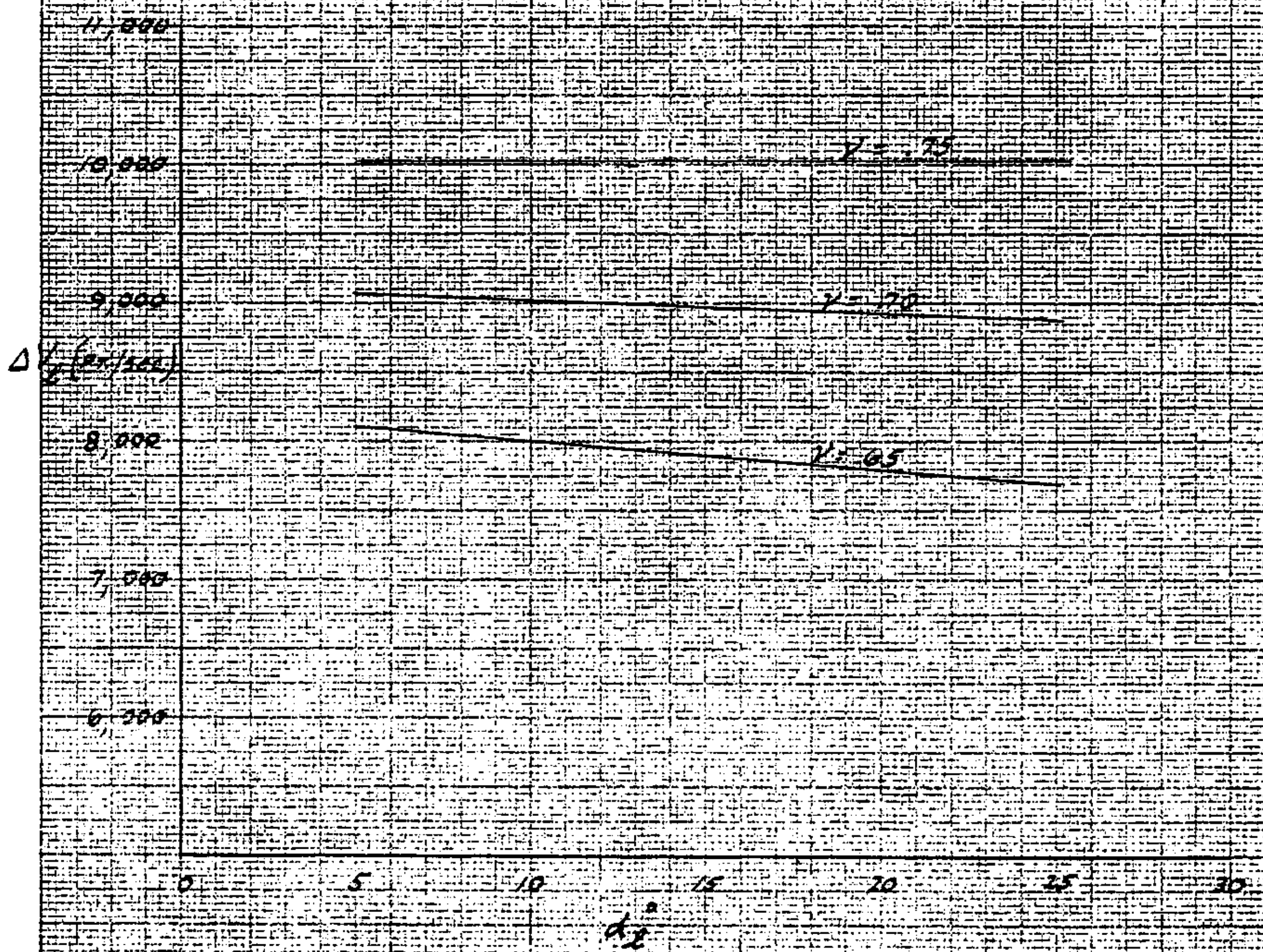


FIGURE 8
APPENDIX II

HYDROGEN-OXYGEN ROCKET

FIRST BURNING PERIOD

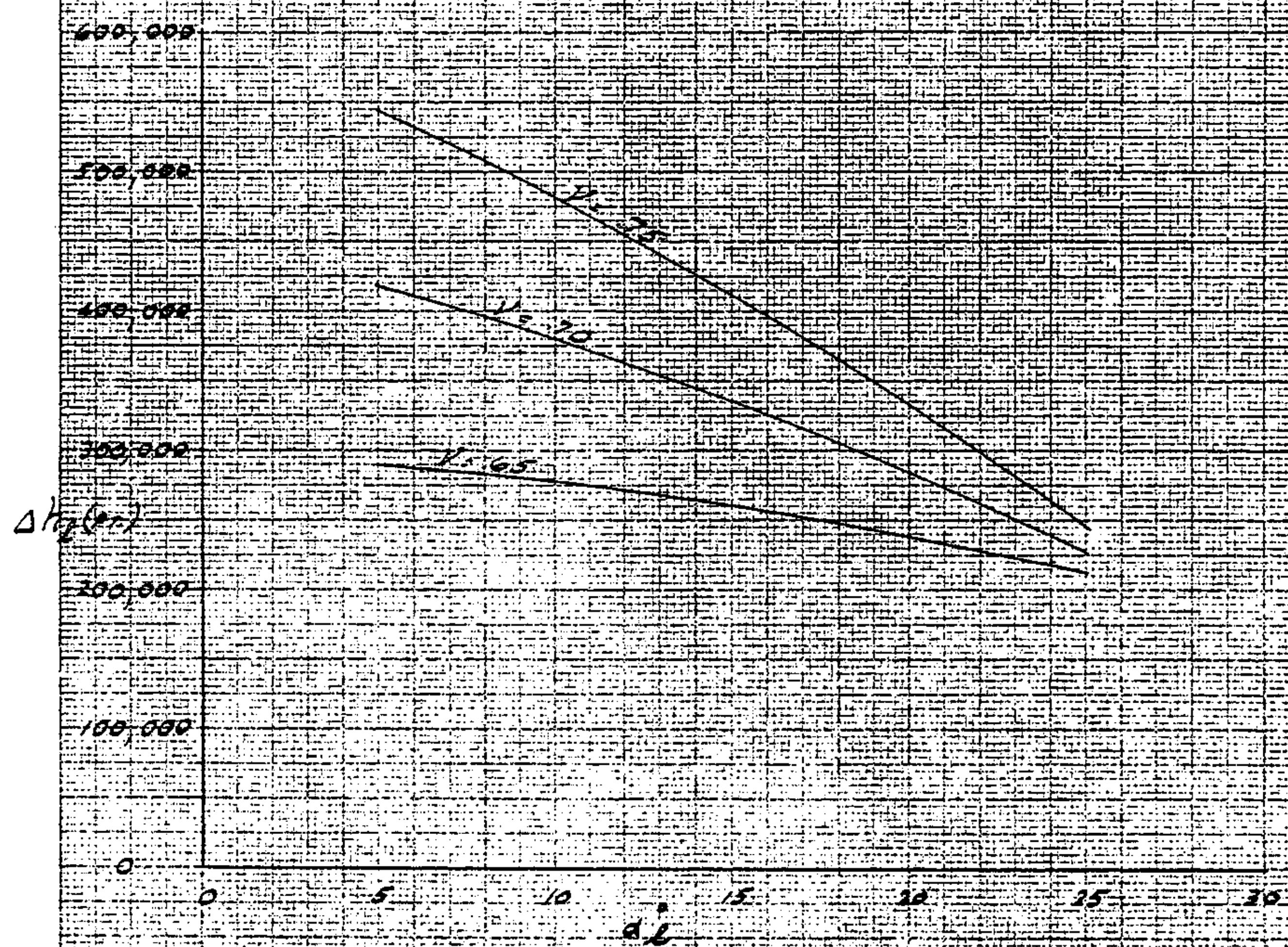


FIGURE 9
APPENDIX D

FORM 25 BP

Analysis PRELIMINARY DESIGN OF SATELLITE VEHICLE

Prepared by W. H. Wampler

Date May 2, 1946

DOUGLAS AIRCRAFT COMPANY, INC.

Model #1033

Report No. SM11827

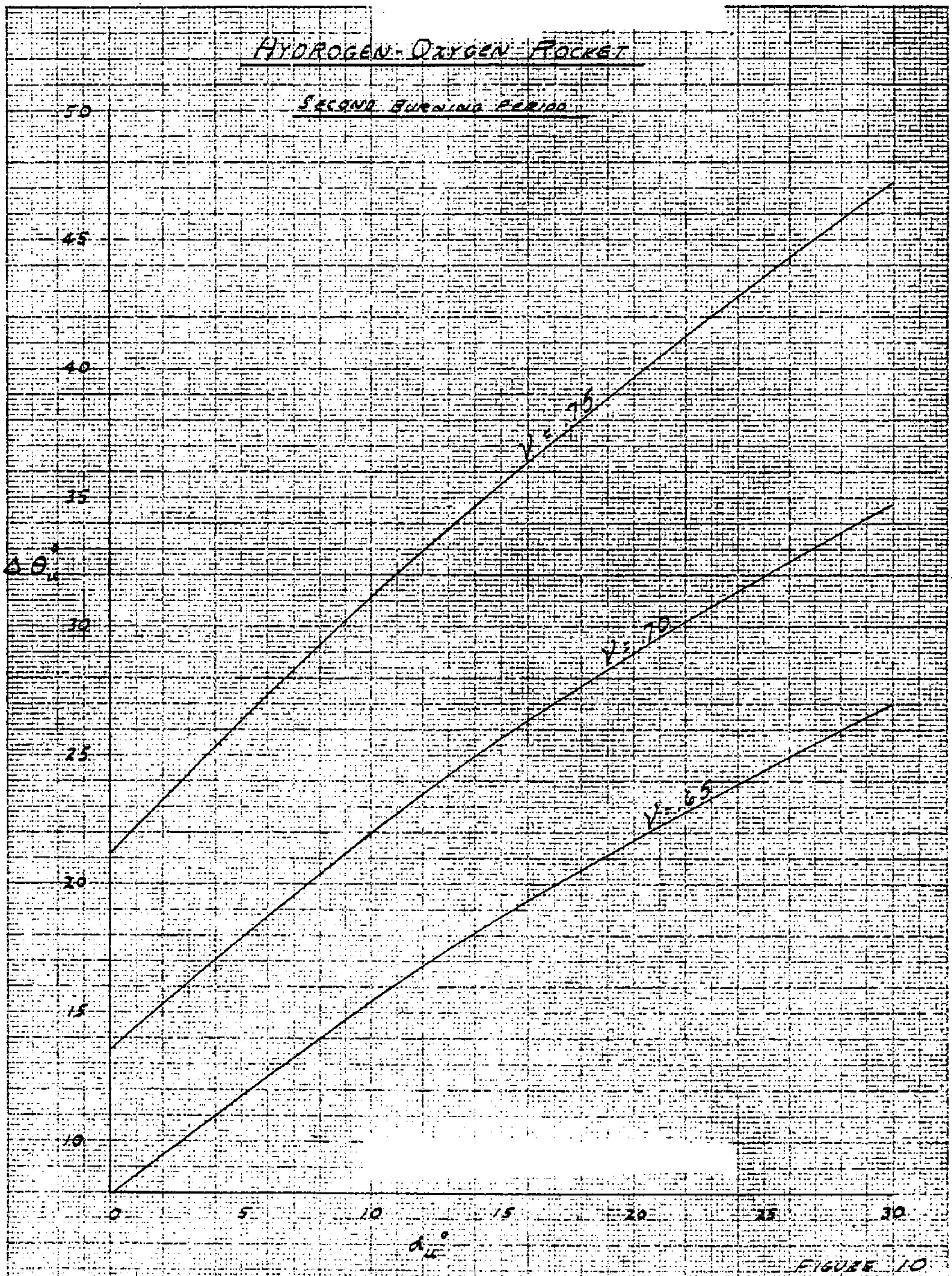


FIGURE 10

APPENDIX D

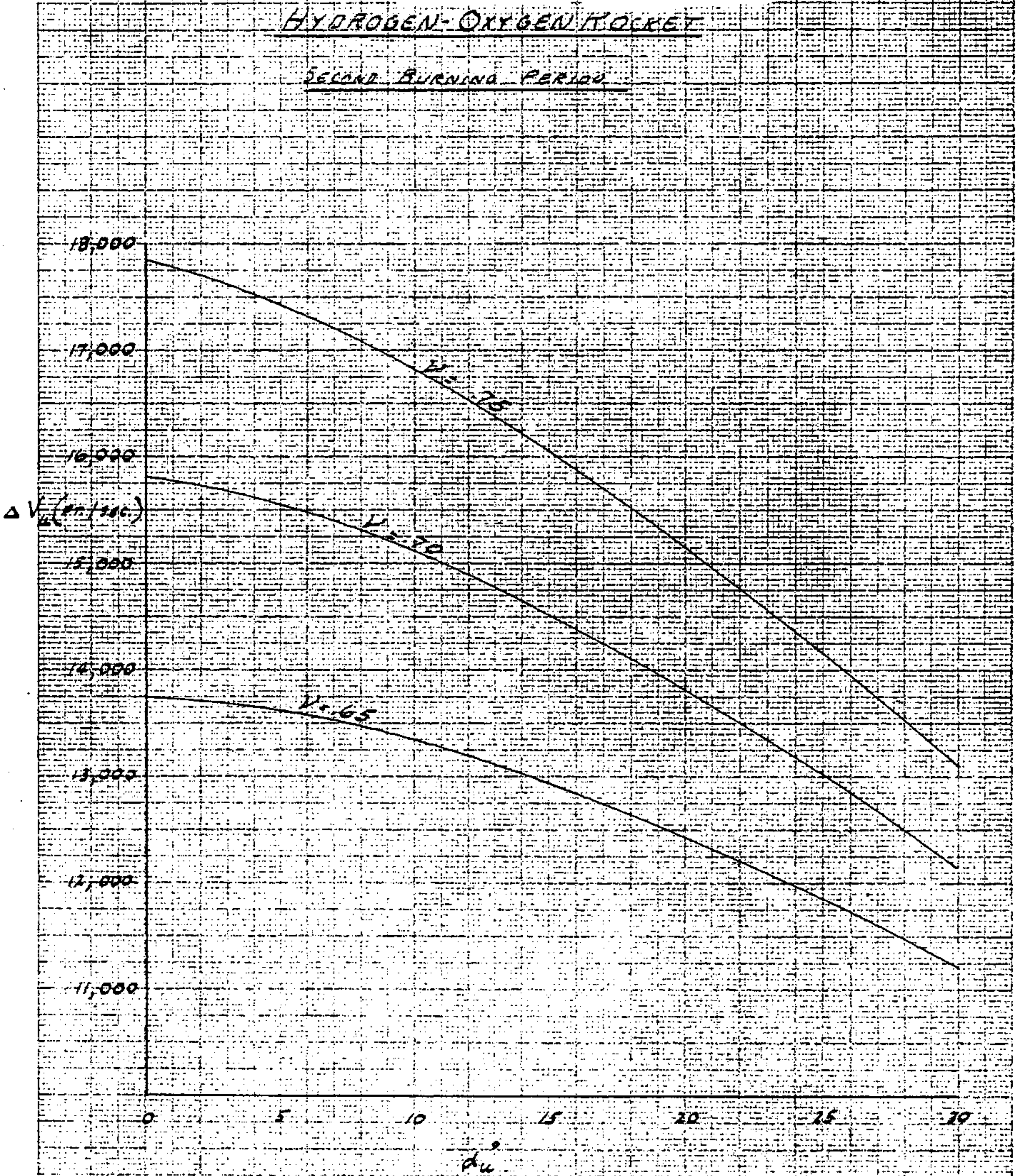


FIGURE 11
APPENDIX D

FORM 25 BP

Analysis PRELIMINARY DESIGN OF SATELLITE VEHICLE

Prepared by W. H. Wampler

Date May 2, 1946

DOUGLAS AIRCRAFT COMPANY, INC.

Model #1033

Report No. SM-11827

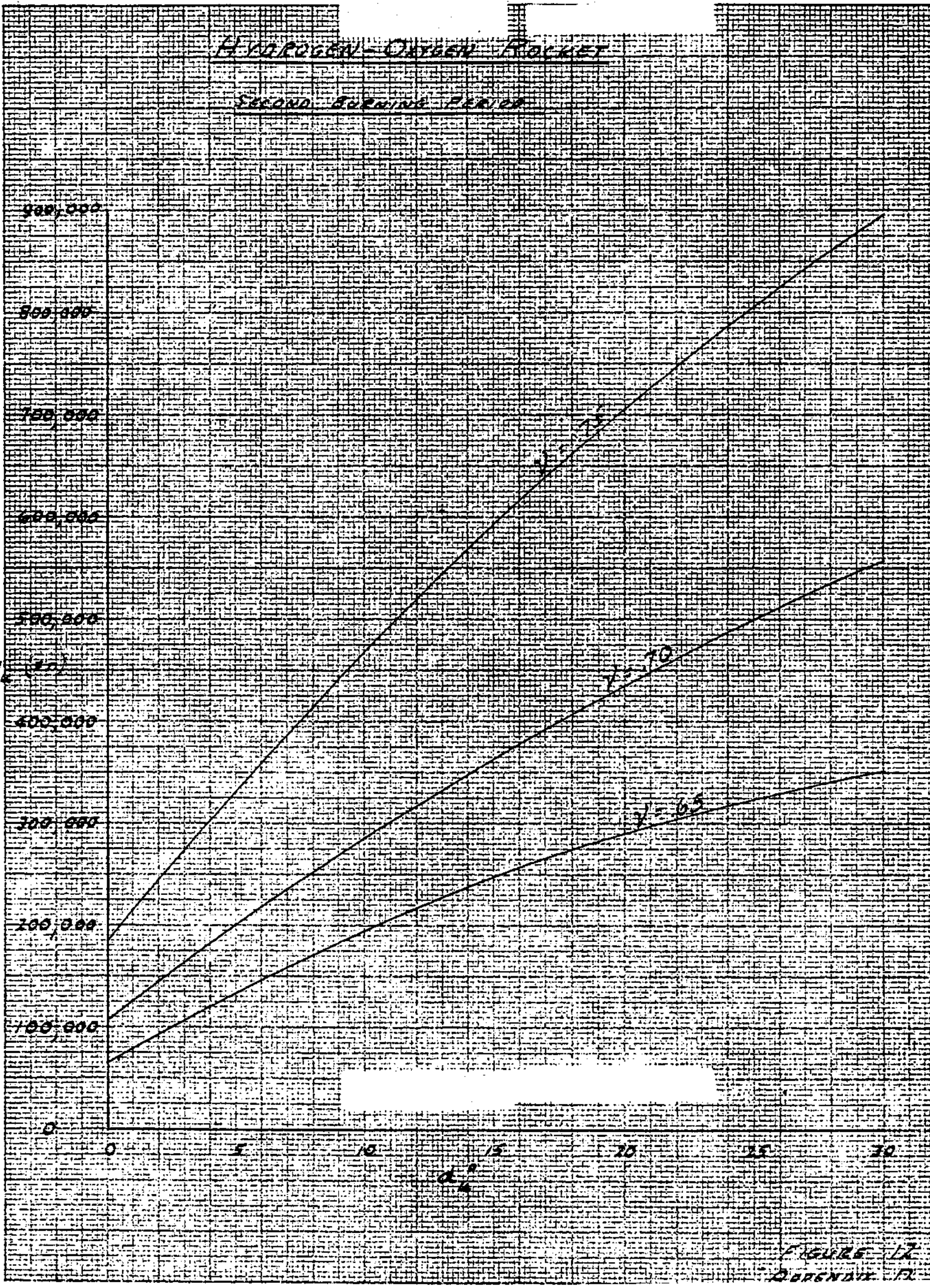


FIGURE 17

APPENDIX II

WORK SHEET
FORM 30-103-1 (REV. 1-43)

1st Burning Period

$\nu = .55$ $\alpha = 15°$

t (sec.)	δt	$1-\nu^{t}/t_B$	$\log(1-\nu^{t}/t_B)$	$\log\frac{1-\nu^{t_2}/t_B}{1-\nu^{t_1}/t_B}$	I	$\bar{I}$	$\nu\bar{I}\log\frac{1-\nu^{t_2}/t_B}{1-\nu^{t_1}/t_B}$	$-g\delta t$	$-g\frac{\bar{D}}{W}\delta t$	δV	Correction due to α	V	$\delta t(V_1+g\bar{I})$	$\frac{g\bar{I}t_B}{\nu}(1-\nu^{t}/t_B)_{t_2} \times \log\frac{1-\nu^{t_2}/t_B}{1-\nu^{t_1}/t_B}$	$1+\frac{D}{W}$
0		1.0000	0		215.0						0	0			
	5			-.0576		215.4	400.	-161	- 2	237			34680	-33668	1.0111
5		.9440	-.0576		215.8						0	237			
	5			-.0612		217.2	428.	-161	-10	257			36155	-33931	1.0597
10		.8880	-.1188		218.6						0	494			
	5			-.0651		220.8	462.	-161	-26	275			38020	-34378	1.1602
15		.8320	-.1839		223.0						0	769			
	5			-.0697		225.9	507.	-161	-64	282			40215	-35122	1.3895
20		.7760	-.2536		228.8						0	1051			
	5			-.0749		232.2	559.	-161	-98	300			42640	-35996	1.6237
25		.7200	-.3285		235.6						19	1332.			
	5			-.0810		239.0	623.	-161	-106	357			45235	-36951.	1.6672
30		.6640	-.4095		242.4						39	1669			
	5			-.0882		245.7	698.	-161	-103	434			47100	-37884	1.6390
35		.6079	-.4977		249.0						59	2083			
	5			-.0967		251.9	785	-161	-98	526			51265	-38645	1.6067
40		.5519	-.5944		254.8						78	2590			
	5			-.1070		257.0	886	-161	-82	643			54715	-39199	1.5002
45		.4959	-.7014		259.2						95	3216			
	4.1			-.0971		260.1	813.	-132	-46	635			47913	-32670	1.3570
49.1		.4500	-.7985		261.0						104	3842			

WORK SHEET
FORM 30-103-1 (REV. 1-43)

1st BURNING PERIOD

$\nu = .55$ $\alpha = 15°$

δt	$-g\frac{(\delta t)^2}{2}(1+\frac{\bar{D}}{W})$	δh	Correction due to θ	h	p	C_D	$\frac{pC_D(V/1000)^2}{1-V/V_{10}}$	D/W	$\bar{D}/W$
0			0	0		0		0	
	-407	605							.0111
5			0	605	2080	.3	37.129	.0222	
	-427	1777							.0597
10			0	2382	1970	.3	162.42	.0972	
	-467	3175							.1602
15			0	5557	1750	.3	373.15	.2232	
	-559	4534							.3895
20			0	10091	1450	.45	928.80	.5558	
	-654	5990							.6237
25			850	15231	1140	.4	1156.0	.6916	
	-671	7613							.6672
30			1749	21945	810	.3033	1074.3	.6428	
	-660	9556							.6390
35			2644	30606	530	.2654	1061.7	.6352	
	-647	11973							.6072
40			3498	41725	310	.24215	968.2	.5792	
	-604	14912							.5002
45			4258	55877	140	.2274	703.8	.4212	
	-367	14876							.3570
49.1			4664	70347	64.5	.21927	489.38	.2928	

WORK SHEET
FORM 30-103-1 (REV. 1-43)

1st Burning Period

$\nu = .55 \qquad \alpha = 15°$

t	$\bar{\theta}$	$\cos\bar{\theta}$	$-\frac{g\delta t}{\bar{V}}\cos\bar{\theta}$	$\frac{\bar{V}\cos\bar{\theta}\,\delta t}{R}$	$\frac{2V_e}{R}\,\delta t$	$\log(1-\nu\frac{t}{t_b})$	$\log\left(\frac{1-\nu t_2/t_b}{1-\nu t_1/t_b}\right)$	$\frac{\bar{I}g\sin\alpha}{V}$	$\log\left(\frac{1-\nu t_2/t_b}{1-\nu t_1/t_b}\right)\times\frac{\bar{I}g\sin\alpha}{V}$	$\delta\theta$	$\theta_{RAD.}$	$\theta°$
24.55						-.3216					1.571	90.00
	1.5755	-.0047	.0001	0	.0001		-.0069	1.4272	-.0098	-.010		
25						-.3285					1.561	89.45
	1.5156	.0552	-.0058	0	.0007		-.0810	1.3020	-.1055	-.110		
30						-.4095					1.451	83.14
	1.4072	.1629	-.0136	0	.0007		-.0882	1.0636	-.0938	-.107		
35						-.4977					1.344	77.03
	1.3031	.2645	-.0177	0	.0007		-.0967	.7728	-.0844	-.101		
40						-.5944					1.243	71.22
	1.2050	.3577	-.0192	.0002	.0007		-.1070	.7163	-.0766	-.095		
45						-.7014					1.148	65.79
	1.1211	.4345	-.0158	.0003	.0005		-.0971	.5974	-.0580	-.073		
49.1						-.7985					1.075	61.60

WORK SHEET
FORM 30-103J (REV. 1-43)

2ND BURNING PERIOD
$\nu = .55$ $\alpha = 15°$

INTERVALS	t	$1-\nu t/t_b$	$\log(1-\nu t/t_b)$	$\log \frac{1-\nu t_2/t_b}{1-\nu t_1/t_b}$	$-g\tau \cos\alpha \times \log \frac{1-\nu t_2/t_b}{1-\nu t_1/t_b}$	$\bar{\theta}$	$\sin\bar{\theta}$	$-g\tau \sin\bar{\theta}$	P	C_D	$\frac{PC_D(\cdot)}{1-\nu t/t_b}$	D/W	$\bar{D}/W$	$-g\tau \frac{\bar{D}}{W}$	δV	V
0	0	1.0000	0				.8841		64.5	.2193	208.78	.126				3842
				-.0566	493.	1.0586	.8715	-143.					.103	-18	332	4008
1	5.085	.9450	-.0566				.8586		34.0	.2173	135.90	.080				4174
				-.0599	522.	1.0072	.8453	-138.					.0645	-10	374	4362
2	10.170	.8900	-.1165				.8316		16.5	.2145	82.27	.049				4548
				-.0638	556.	.9571	.8175	-134.					.040	-6	416	4756
3	15.255	.8350	-.1803				.8069		8.5	.2122	53.23	.031				4964
				-.0682	594.	.9080	.7883	-129.					.027	-4	462	5195
4	20.340	.7800	-.2485				.7731		4.7	.2102	37.29	.023				5426
				-.0731	637.	.8601	.7578	-124.					.019	-3	510	5681
5	25.425	.7250	-.3216				.7423		2.27	.2085	23.01	.014				5936
				-.0789	687.	.8135	.7267	-119.					.011	-2	566	6219
6	30.510	.6700	-.4005				.7107									6502
				-.0856	745.	.7681	.6947	-114.						-1	630	6817
7	35.595	.6150	-.4861				.6786									7132
				-.0937	816.	.7240	.6624	-109.						0	707	7486
8	40.680	.5600	-.5798				.6459									7839
				-.1034	901.	.6809	.6295	-103.							798	8238
9	45.765	.5050	-.6832				.6107									8637
				-.1153	1004.	.6388	.5962	-98.							906	9090
10	50.850	.4500	-.7985				.5794									9543

WORK SHEET
FORM 30-103-1 (REV. 1-43)

Page

2ND BURNING PERIOD

$\nu = .55$ $\alpha = 15°$

INTERVALS	Cos θ	$-\frac{g\delta t\cos\theta}{V}$	$\frac{V\cos\theta}{R}\delta t$	$\frac{2V_E}{R}\delta t$	$\frac{g\Sigma\sin\theta}{V}$ × log [illegible]	dθ	θ rad.	θ°	V sin θ	dh	h
0							1.0846	62.143	3396		70347
	.4904	-.0200	.0004	.0007	-.0330	-.0520			3490	17747	
1							1.0326	59.16	3584		88094
	.5342	-.0200	.0006	.0007	-.0320	-.0507			3683	18728	
2							.9819	56.26	3782		106822
	.5760	-.0198	.0007	.0007	-.0313	-.0497			3884	19750	
3							.9322	53.41	3986		126572
	.6153	-.0194	.0008	.0007	-.0306	-.0485			4090	20798	
4							.8837	50.63	4194		147370
	.6524	-.0188	.0009	.0007	-.0300	-.0472			4300	21866	
5							.8365	47.93	4406		169236
	.6870	-.0181	.0010	.0007	-.0296	-.0460			4513	22949	
6							.7905	45.29	4620		192185
	.7193	-.0173	.0012	.0007	-.0293	-.0447			4730	24052	
7							.7458	42.73	4840		216237
	.7492	-.0164	.0013	.0007	-.0292	-.0436			4952	25181	
8							.7022	40.23	5063		241418
	.7770	-.0155	.0015	.0007	-.0293	-.0426			5169	26284	
9							.6596	37.79	5275		267702
	.8028	-.0145	.0018	.0007	-.0296	-.0416			5402	27469	
10							.6180	35.41	5529		295171

WORK SHEET
FORM 30-103-1 (REV. 1-43)

3RD BURNING PERIOD
$\nu = .55$ $\alpha = 15°$

INTERVALS	t	$1-\nu^{t}/_{t_B}$	$\log(1-\nu^{t}/_{t_B})$	$\log\frac{1-\nu^{t_2}/_{t_B}}{1-\nu^{t_1}/_{t_B}}$	$-gI\cos\alpha \times \log\frac{1-\nu^{t_2}/_{t_B}}{1-\nu^{t_1}/_{t_B}}$		Sin θ	$-g\delta t \sin\theta$	δV	V		V Sin θ	δh	h
0							.5794			9543		5529		295171
							.5693	-93	400	9743		5544	28191	
1							.5590			9943		5559		323362
							.5438	-89.	433	10159		5572	28334	
2							.5384			10376		5586		351696
							.5280	-87.	469	10611		5600	28476	
3							.5177			10845		5614		380172
							.5074	-83.	511	11101		5630	28629	
4		SAME VALUES AS					.4971			11356		5646		408201
							.4868	-80.	557	11635		5659	28776	
5		IN 2ND BURNING PERIOD					.4762			11913		5672		437577
							.4657	-76.	611	12219		5686	28913	
6							.4552			12524		5700		466490
							.4447	-73.	672	12860		5715	29061	
7							.4342			13196		5730		495551
							.4235	-69.	747	13570		5744	29208	
8							.4130			13943		5757		524759
							.4024	-66.	835	14361		5772	29351	
9							.3915			14778		5786		554110
							.3807	-62.	942	15249		5800	29493	
10							.3699			15720		5814		583603

WORK SHEET
FORM 30-103-1 (REV. 1-43).

3RD BURNING PERIOD
$\nu = .55$ $\alpha = 15°$

INTERVALS	$\cos\theta$	$-\frac{g\,\delta t \cos\theta}{V}$	$\frac{V\cos\theta}{R}\,\delta t$	$\frac{2V_E}{R}\,\delta t$	$\frac{g\,I\,\sin\alpha}{V}$ $n \log \frac{1-\nu^{n}/10}{1-\nu^{n}/10}$	$\delta\theta$	$\theta_{RAD.}$	$\theta°$
0							.6180	35.41
	.8222	-.0138	.0019	.0007	-.0136	-.0248	.6056	
1							.5932	33.99
	.8392	-.0135	.0020	.0007	-.0137	-.0247	.5749	
2							.5625	32.57
	.8493	-.0131	.0021	.0007	-.0140	-.0243	.5563	
3							.5442	31.18
	.8617	-.0127	.0023	.0007	-.0143	-.0240	.5322	
4							.5202	29.81
	.8735	-.0123	.0024	.0007	-.0147	-.0239	.5083	
5							.4963	28.44
	.8850	-.0119	.0026	.0007	-.0151	-.0237	.4844	
6							.4726	27.02
	.8957	-.0114	.0027	.0007	-.0155	-.0235	.4608	
7							.4491	25.73
	.9059	-.0109	.0029	.0007	-.0161	-.0234	.4374	
8							.4257	24.39
	.9155	-.0105	.0032	.0007	-.0168	-.0234	.4140	
9							.4023	23.05
	.9247	-.0099	.0034	.0007	-.0176	-.0234	.3906	
10							.3789	21.71

WORK SHEET
FORM 30-103-1 (REV. 1-43)

4TH BURNING PERIOD
$\nu = .55$ $\alpha = 30^\circ$

t	ν^{t/t_B}	$1-\nu^{t/t_B}$	$\log(1-\nu^{t/t_B})$	$\log\frac{1-\nu^{t_1/t_B}}{1-\nu^{t_2/t_B}}$	$-gI\cos\alpha \times \log\frac{1-\nu^{t_1/t_B}}{1-\nu^{t_2/t_B}}$		$\sin\theta$	$-g\delta t\sin\theta$	δV	V		$V\sin\theta$	δh	h
45.6	.55	.45	-.798				0			24560		0		0
				-.476	3710		.0494	-36	3674			1100	25100	
22.8	.275	.725	-.322				.0985			20886		2059		25100
				-.322	2508		.1413	-104	2404			2800	64000	
0	0	1.000	0				.1840			18482		3400		89100

WORK SHEET
FORM 30-103-1 (REV. 1-43)

4TH BURNING PERIOD
$\nu = .55$ $\alpha = 30°$

t		$\cos\theta$	$-\frac{g\cos\theta}{V}$	$\frac{V}{R}\cos\theta$	$-\frac{g\cos\theta}{V} + \frac{V}{R}\cos\theta + \frac{2V_0}{R}$	$\Delta t\left(-g\frac{\cos\theta}{V} + \frac{V}{R}\cos\theta + \frac{2V_0}{R}\right)$	$\frac{gI\sin\alpha}{V} \times \log\frac{1-\nu^{t_1/t_0}}{1-\nu^{t_2/t_0}}$	$d\theta$	θ RAD.	$\theta°$
45.6									0	0
		.999	-.00141	.00107	-.00019	-.0044	-.0943	-.0987		
22.8									.0987	5.7
		.990	-.00162	.00092	-.00055	-.0125	-.0737	-.0862		
0									.1849	10.6

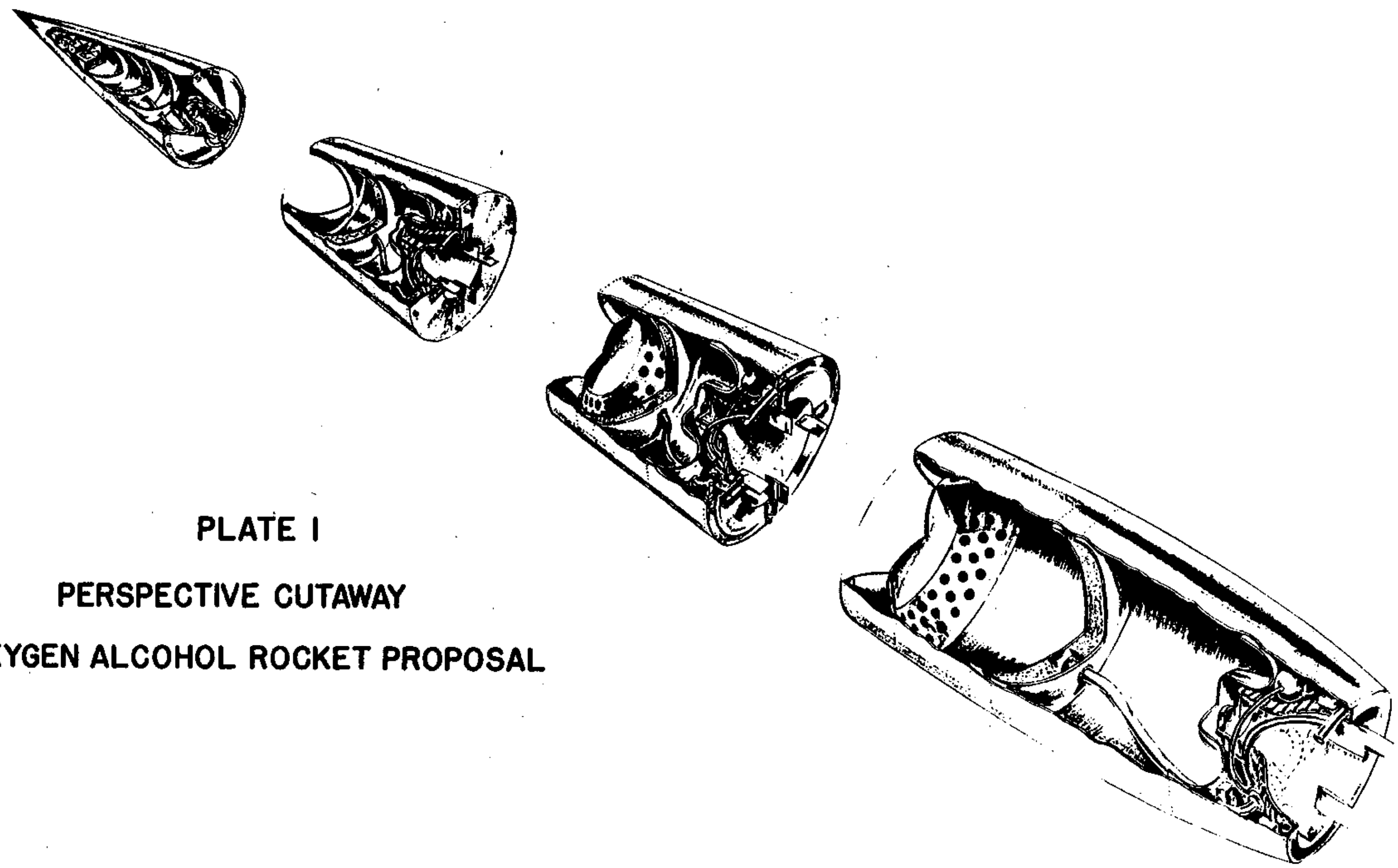


PLATE I

PERSPECTIVE CUTAWAY

OXYGEN ALCOHOL ROCKET PROPOSAL

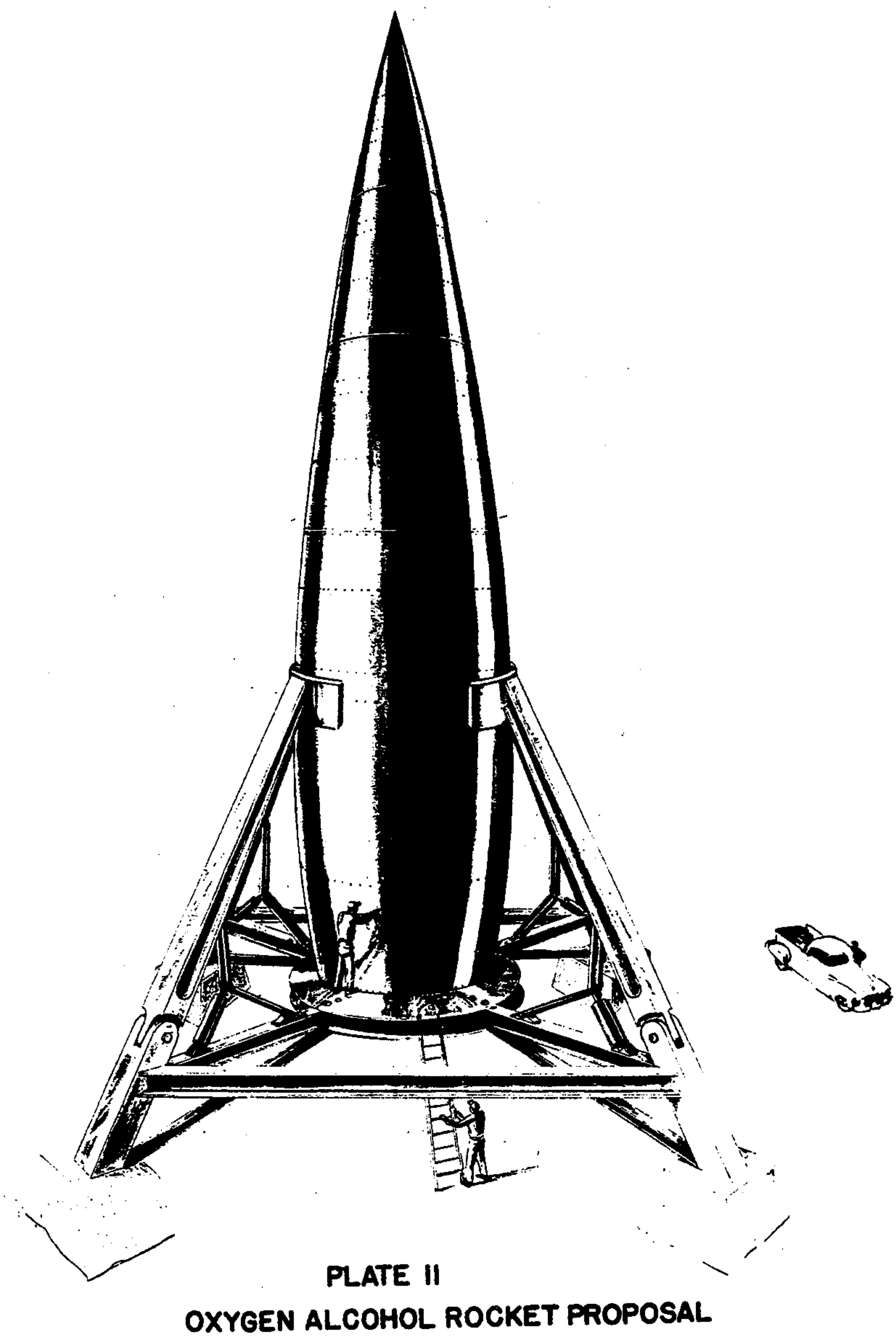


PLATE II

OXYGEN ALCOHOL ROCKET PROPOSAL
ASSEMBLED FOR FIRING

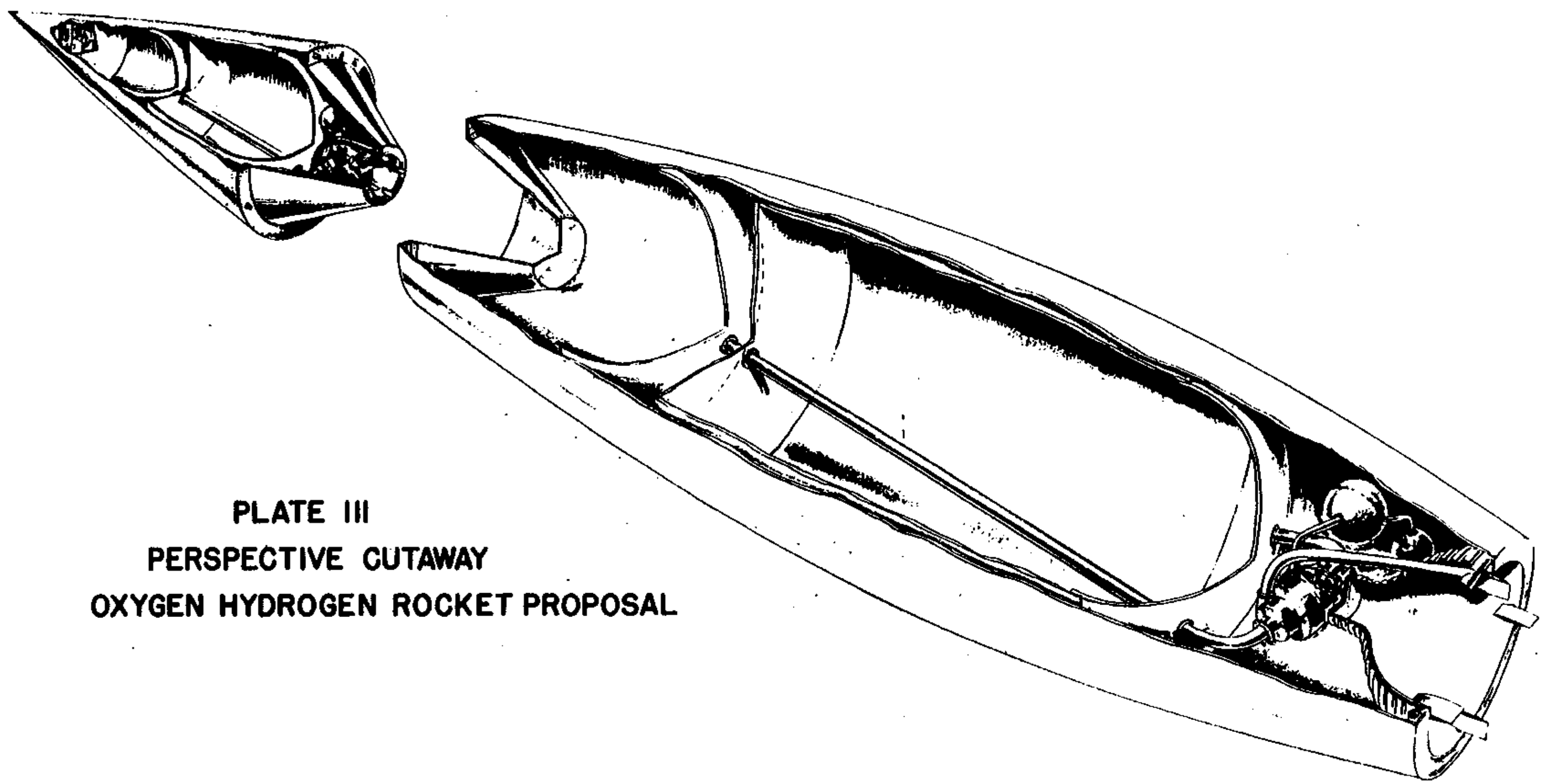


PLATE III

PERSPECTIVE CUTAWAY

OXYGEN HYDROGEN ROCKET PROPOSAL

APPENDIX E

E. Development of Small Perturbation Equations of Motion.

From Appendix C page 2 we have the equations

$$m\ddot{r} - mr(\dot{\phi}+\Omega)^2 + \frac{k\,m\,M}{r^2} = F_r$$

$$2m r\dot{r}(\dot{\phi}+\Omega) + mr^2\ddot{\phi} = rF_\phi$$

Referring to a stationary coordinate system the equations become

$$r^2\ddot{r} - r^3\dot{\phi}^2 + kM = \frac{F_r r^2}{m}$$

$$2\dot{r}\dot{\phi} + r\ddot{\phi} = \frac{F_\phi}{m}$$

Let $r = r_0 + \Delta r$ and $\dot{\phi} = \omega_0 + \Delta\omega$

Then $\left[r_0^2 + 2r_0(\Delta r) + (\Delta r)^2\right](\Delta\ddot{r}) - \left[r_0^3 + 3r_0^2(\Delta r) + 3r_0(\Delta r)^2 + (\Delta r)^3\right]\left[\omega_0^2 + 2\omega_0(\Delta\omega) + (\Delta\omega)^2\right]$

$$+ kM = \frac{F_r}{m}\left[r_0^2 + 2r_0(\Delta r) + (\Delta r)^2\right]$$

and $2(\Delta\dot{r})\left[\omega_0 + (\Delta\omega)\right] + \left[r_0 + (\Delta r)\right](\Delta\dot{\omega}) = \frac{F_\phi}{m}$

$$r_0^2(\Delta\ddot{r}) - r_0^3\omega_0^2 + kM - \frac{F_r}{m}r_0^2 + (\Delta r)\left[2r_0(\Delta\ddot{r}) - 3r_0^2\omega_0^2 - \frac{2r_0F_r}{m}\right]$$

$+(\Delta\omega)\left[-2\omega_0 r_0^3\right] \doteq 0$ neglecting higher order terms

and $2(\Delta\dot{r})\omega_0 + r_0(\Delta\dot{\omega}) \doteq \frac{F_\phi}{m}$

If the departure from a circular orbit is small the coefficient of Δr in the first equation becomes $-3r_0^2\omega_0^2$. Also $kM = g_0 r_0^2$.

$$(\Delta\ddot{r}) + (g_0 - r_0\omega_0^2) - \frac{F_r}{m} - 3\omega_0^2(\Delta r) - 2r_0\omega_0(\Delta\omega) \doteq 0$$

$$2(\Delta\dot{r})\omega_0 + r_0(\Delta\dot{\omega}) - \frac{F_\phi}{m} \doteq 0$$

To eliminate $(\Delta\omega)$

$$(\Delta\omega) = \frac{(\Delta\ddot{r}) + (g_0 - r_0\omega_0^2) - \frac{F_r}{m} - 3\omega_0^2(\Delta r)}{2r_0\,\omega_0}$$

$$(\Delta\dot{\omega}) = \frac{(\Delta\dddot{r}) - \left(\frac{\dot{F}_r}{m}\right) - 3\omega_0^2(\Delta\dot{r})}{2r_0\,\omega_0}$$

$$2(\Delta\dot{r})\omega_0 + \frac{(\Delta\dddot{r}) - \left(\frac{\dot{F}_r}{m}\right) - 3\omega_0^2(\Delta\dot{r})}{2\omega_0} - \frac{F_\phi}{m} \doteq 0$$

$$(\Delta\dddot{r}) + (\Delta\dot{r})\,\omega_0^2 - \left(\frac{\dot{F}_r}{m}\right) - \frac{F_\phi}{m}2\omega_0 \doteq 0$$

OR $\frac{d^2V_R}{dt^2} + \omega_0^2 V_R = \frac{2\omega_0 F_\phi}{m} + \frac{d}{dt}\left(\frac{F_r}{m}\right)$ WHERE $V_R = (\Delta\dot{r})$

LET $V_R = A\cos(\omega_0 t - \delta) + B$ (NEGLECTING $\frac{d}{dt}\left(\frac{F_r}{m}\right)$ AND

THEN $B = \frac{2F_\phi}{\omega_0 m}$ ASSUMING $\frac{F_\phi}{m}$ CONSTANT)

$$V_R = A\cos(\omega_0 t - \delta) + \frac{2F_\phi}{\omega_0 m}$$

$$\Delta r = A\int_0^t \cos(\omega_0 t - \delta)\,dt + \frac{2}{\omega_0}\int_0^t \frac{F_\phi}{m}\,dt$$

$$\left(\frac{\Delta r}{r_0}\right) \text{ PER REVOLUTION} = \frac{A}{r_0}\int_0^{\frac{2\pi}{\omega_0}} \cos(\omega_0 t - \delta)\,dt + \frac{2}{\omega_0 r_0}\int_0^{\frac{2\pi}{\omega_0}} \frac{F_\phi}{m}\,dt$$

$$\left(\frac{\Delta r}{r_0}\right) \text{ PER REVOLUTION} = 4\pi\left(\frac{F_\phi}{W_0}\right)\frac{g_0}{r_0\,\omega_0^2} \quad \text{FOR } \frac{F_\phi}{m} \text{ CONSTANT}$$

BUT $g_0 \doteq r_0\,\omega_0^2$

AND $\left(\frac{\Delta r}{r_0}\right)$ PER REVOLUTION $= 4\pi\left(\frac{F_\phi}{W}\right)$

Appendix F

F. ORBIT CALCULATION

Orbital Motion Under the Newtonian Law* - In this appendix, the equations of motion of a body will be developed in a form suitable for use in calculating the trajectory of a body after it has been accelerated to the proper speed and direction for orbital motion.

The body is treated as a particle of unit mass acting under a central force varying as the inverse square of the distance from the center of the earth, in accordance with the Newtonian law of universal gravitation.

If (r, ϕ) be the coordinates of the body with respect to the central force, the kinetic energy of the particle is

$$T = 1/2\ (\dot{r}^2 + r^2\dot{\phi}^2).$$

Letting P denote the acceleration directed to the center of force, the work done by the force in an arbitrary infinitesinal displacement, $(dr,\ d\phi)$ is equal to $-Pdr$.

The Lagrangian equations of motion for the particle are

$$\begin{cases} \ddot{r} - r\dot{\phi}^2 = -P, \\ \dfrac{d}{dt}(r^2\dot{\phi}) = 0. \end{cases}$$

The latter of these equations gives on integration

$$r^2\dot{\phi} = H,$$

where H is a constant. This integral corresponds to the ignorable coordinate ϕ, and can be interpreted physically as the integral of angular momentum of the particle about the center of force.

Eliminating dt from the first equation by use of the relationship

* Whittaker, "A Treatise on the Analytical Dynamics of Particles and Rigid Bodies", Cambridge 1937; PP. 86-90.

Appendix F

$$\frac{d}{dt} = \frac{H}{r^2}\frac{d}{d\phi},\quad \text{we obtain}$$

$$\frac{H}{r^2}\frac{d}{d\phi}\left(\frac{H}{r^2}\frac{dr}{d\phi}\right) - \frac{H^2}{r^3} = -P.$$

Letting $u = \frac{1}{r}$, we obtain the differential equation of the path,

$$\frac{d^2u}{d\phi^2} + u = \frac{P}{H^2u^2}. \qquad (1)$$

If the particle be projected from the point whose polar coordinates are (R_o, α) with a velocity V_s in a direction making an angle γ with R_o, the angular momentum is

$$H = R_oV_s \sin\gamma.$$

If the central force per unit mass be μu^2, then

$$P = \mu u^2.$$

Substituting this relation in equation (1) we have

$$\frac{d^2u}{d\phi^2} + u = \frac{\mu}{V_s^2R_o^2\sin^2\gamma}$$

This is a linear differential equation with constant coefficients whose integral is

$$u = \frac{\mu}{V_s^2R_o^2\sin^2\gamma}\left\{1 + e\cos(\phi - \bar{w})\right\},$$

where e and $\bar{w}$ are constants of integration. This is the equation of a conic in polar coordinates whose focus is at the origin, whose eccentricity is e, and whose semi-latus rectum is

$$l = \frac{V_s^2R_o^2\sin^2\gamma}{\mu}$$

The constant $\bar{w}$ determines the position of the apse-line.

Appendix F

Initially we have

$$\phi = \alpha, \; u = \frac{1}{R_o}, \text{ and } \frac{du}{d\phi} = -\frac{\cot\gamma}{R_o}.$$

Hence, it follows that

$$e^2 = 1 + \frac{V_s^4 R_o^2 \sin^2\gamma}{\mu^2} - \frac{2\,V_s^2 R_o \sin^2\gamma}{\mu}, \text{ and}$$

$$\cot(\alpha - \bar{w}) = \frac{-\mu}{R_o V_s^2 \sin\gamma \cos\gamma} + \tan\gamma .$$

The semi-major axis, when conic is an ellipse is generally denoted by a, and is given by

$$a = \frac{\ell}{1-e^2}.$$

Substituting the values of e^2 and ℓ already determined, we have

$$a = \frac{R_o}{2 - \frac{V_s^2 R_o}{\mu}}$$

which determines a in terms of the initial data.

If V_o be the orbital velocity of the particle, then by equating central forces,

$$\frac{\mu m}{r^2} = \frac{m V_o^2}{r}.$$

For $r = R_o$, this gives $V_o^2 = \frac{\mu}{R_o}$. Defining V_s by the relation

$$V_s = V_o \left(1 + \frac{\Delta V}{V_o}\right), \text{ we obtain}$$

$$\frac{V_s^2 R_o}{\mu} = \left(1 + \frac{\Delta V}{V_o}\right)^2 .$$

Appendix F

Making this substitution in the equations previously derived, we have

$$a = \frac{R_o}{2 - \left(1 + \frac{\Delta V}{V_o}\right)^2},$$

$$e^2 = 1 - \sin^2\gamma + 4\left(\frac{\Delta V}{V_o}\right)^2 \sin^2\gamma \left[1 + 1/2 \frac{\Delta V}{V_o}\right]^2, \text{ and}$$

$$\ell = R_o \sin^2\gamma \left(1 + \frac{\Delta V}{V_o}\right)^2.$$

If R_{min} be the minimum radius of the orbit from the center of the earth,

$$e = 1 - \frac{R_{min}}{a} = 1 - \frac{R_{min}}{R_o}\left[2 - \left(1 + \frac{\Delta V}{V_o}\right)^2\right].$$

It will be useful to know the difference between the maximum and minimum distance of the orbit above the earth's surface. Denoting this quantity by ΔH_{max}, we have

$$\begin{aligned} \Delta H_{max} &= 2ae \\ &= 2R_{min}\left(\frac{e}{1-e}\right) \\ &= 2\left[\frac{R_o}{2 - \left(1 + \frac{\Delta V}{V_o}\right)^2} - R_{min}\right]. \end{aligned}$$

If we wish to know the required height at the start of the orbit for given values of ΔH_{max}, R_{min}, and $\frac{\Delta V}{V_o}$, we find

$$R_o = \left[R_{min} + \frac{\Delta H_{max}}{2}\right]\left[2 - \left(1 + \frac{\Delta V}{V_o}\right)^2\right].$$

Letting $\Delta H = R_o - R_{min}$, denote the maximum altitude loss with respect to the starting altitude, we get

$$\Delta H = \left[R_{min} + \Delta H_{max}\right] - \left[R_{min} + \frac{\Delta H_{max}}{2}\right]\left[1 + \frac{\Delta V}{V_o}\right]^2.$$

Appendix F

The following pages contain plots of the relationships between the major characteristics of the orbit.

Analysis ORBIT VEHICLE

Prepared by C. V. STURDEVANT

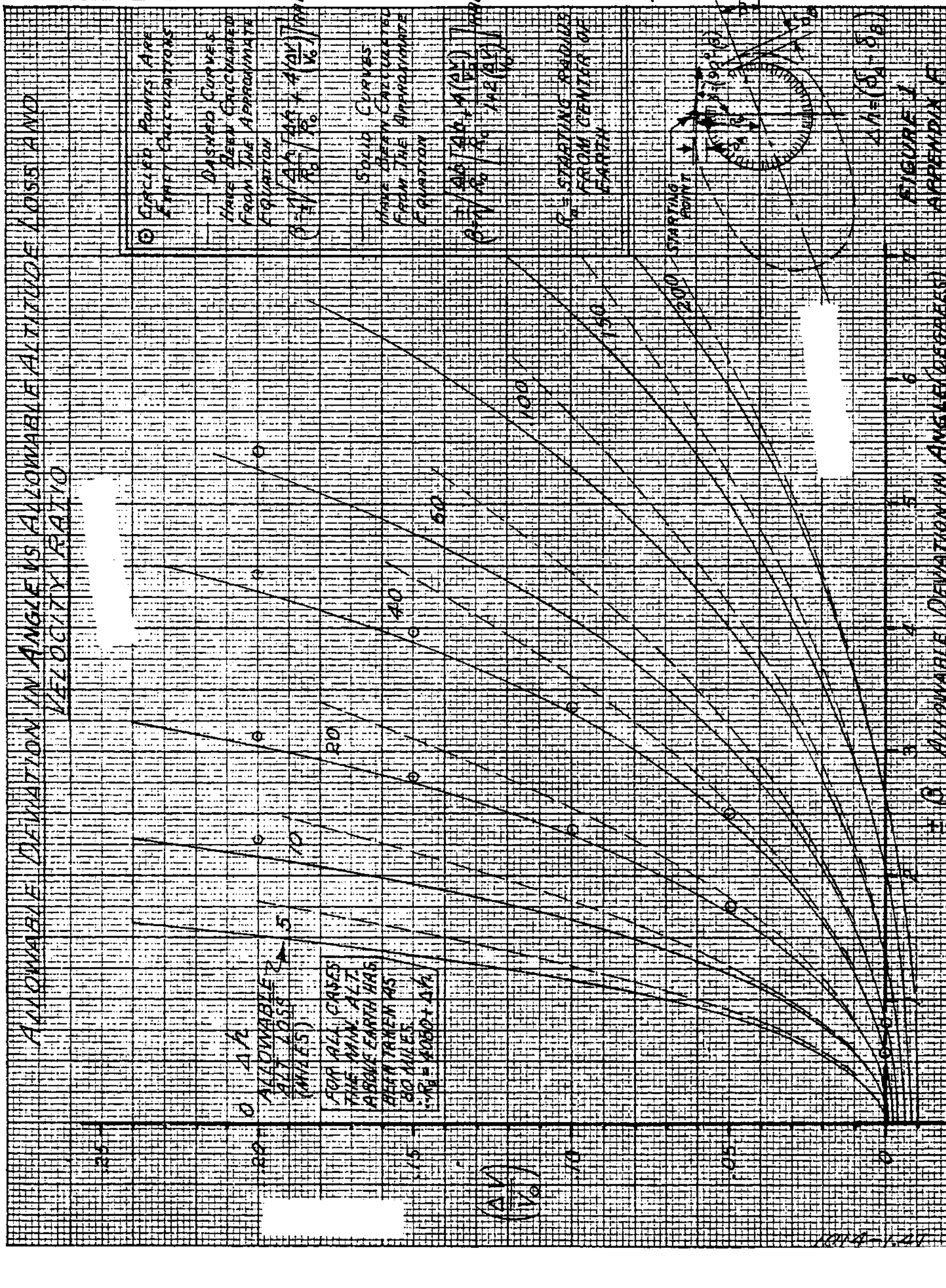
Date 4-26-46

S.M. Plant

Page 6 F

Model #1033

Report No. SM 11827



C.V.S.

1014-1.47

Analysis ORBIT VEHICLE

Prepared by C.V. STURDEVANT

Date 4-24-46

Plant

Model # 1033

Report No. SM 11827

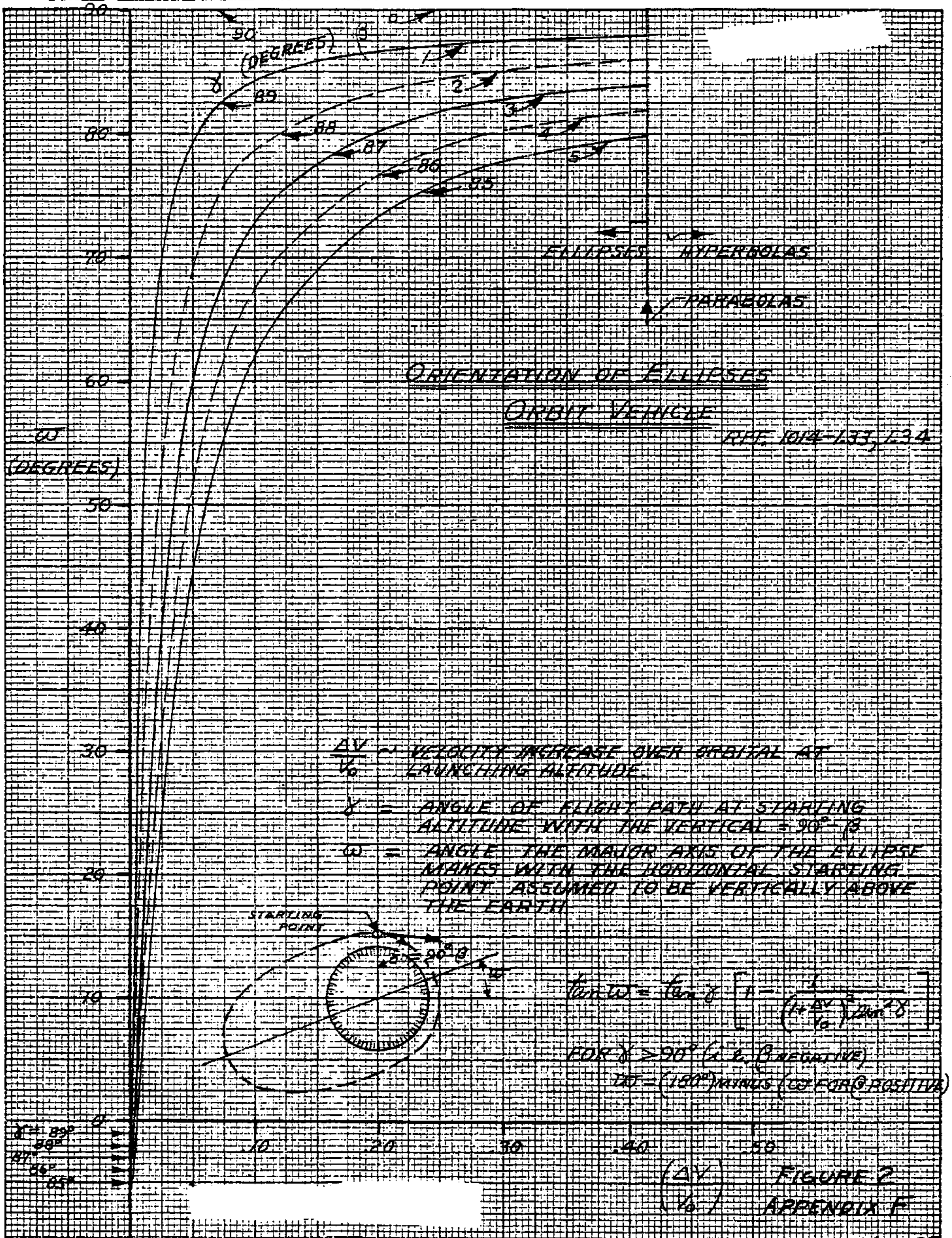
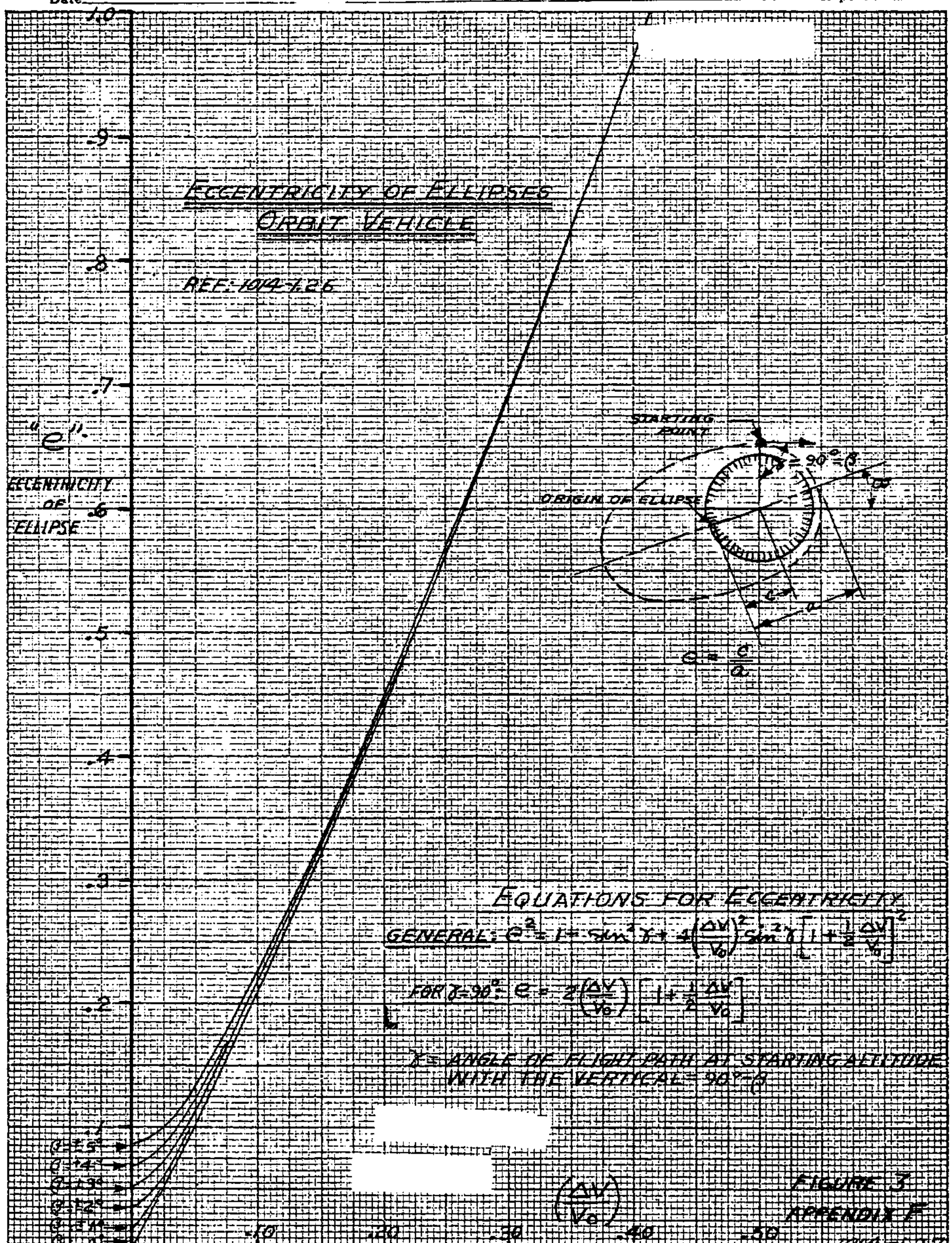


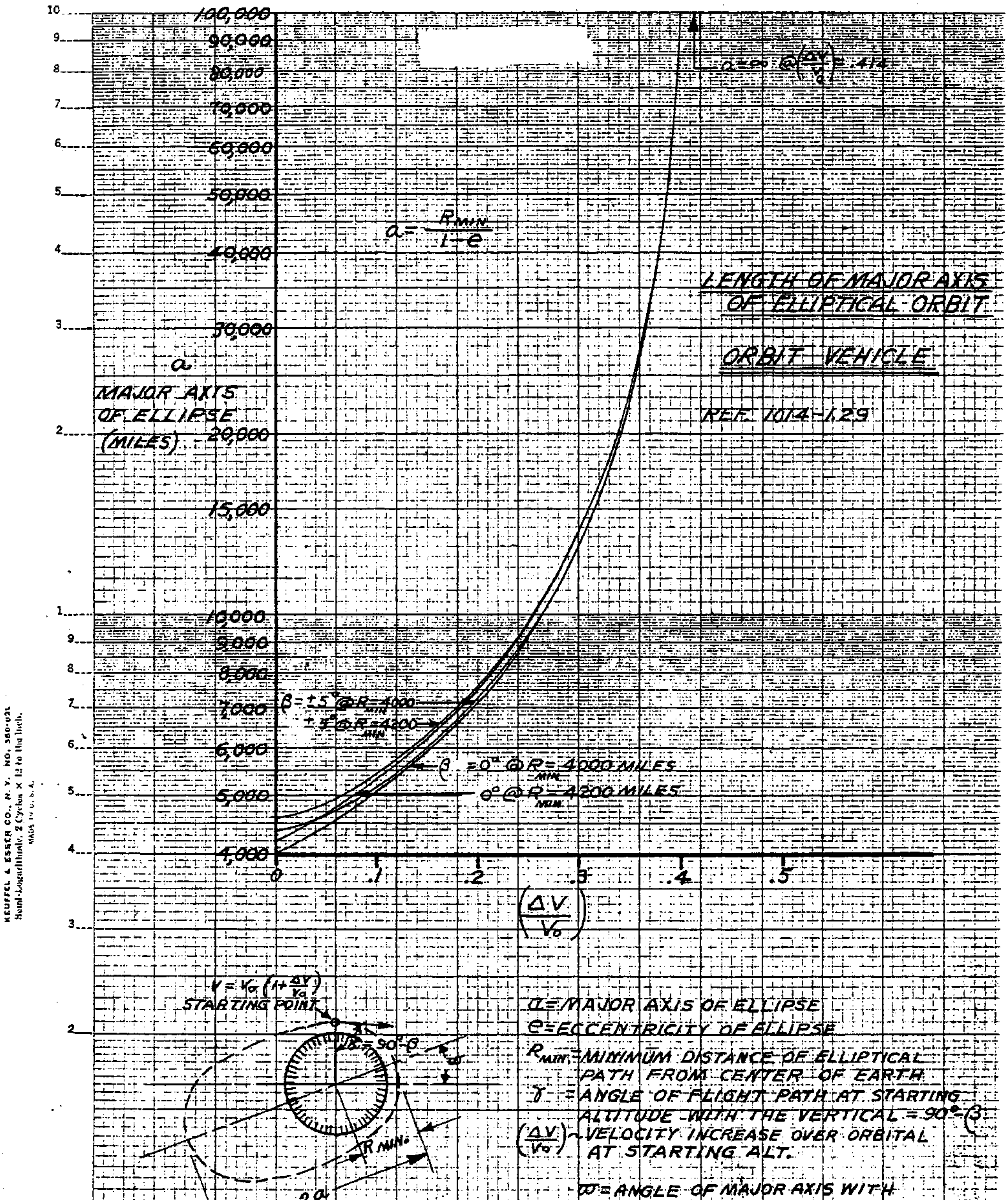
FIGURE 2
APPENDIX F

C.V.S.

1014-138


ECCENTRICITY OF ELLIPSES
ORBIT VEHICLE
REF: 1014-1.26
"e"
ECCENTRICITY OF ELLIPSE
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
STARTING POINT
ORIGIN OF ELLIPSE
= 90° - β
e = c/a
EQUATIONS FOR ECCENTRICITY
GENERAL: e² = 1 - sin² γ + 4(ΔV/V₀)² sin² γ [1 + ½ ΔV/V₀]²
FOR γ = 90°: e = 2(ΔV/V₀) [1 + ½ ΔV/V₀]
γ = ANGLE OF FLIGHT PATH AT STARTING ALTITUDE WITH THE VERTICAL = 90° - β
β = 5°
β = 4°
β = 3°
β = 2°
β = 1°
β = 0
(ΔV/V₀)
.10
.20
.30
.40
.50
FIGURE 3
APPENDIX F
1014-1.27

C.V.S.
M.G.



M.G.

FIGURE 4
APPENDIX F
1014-1.30

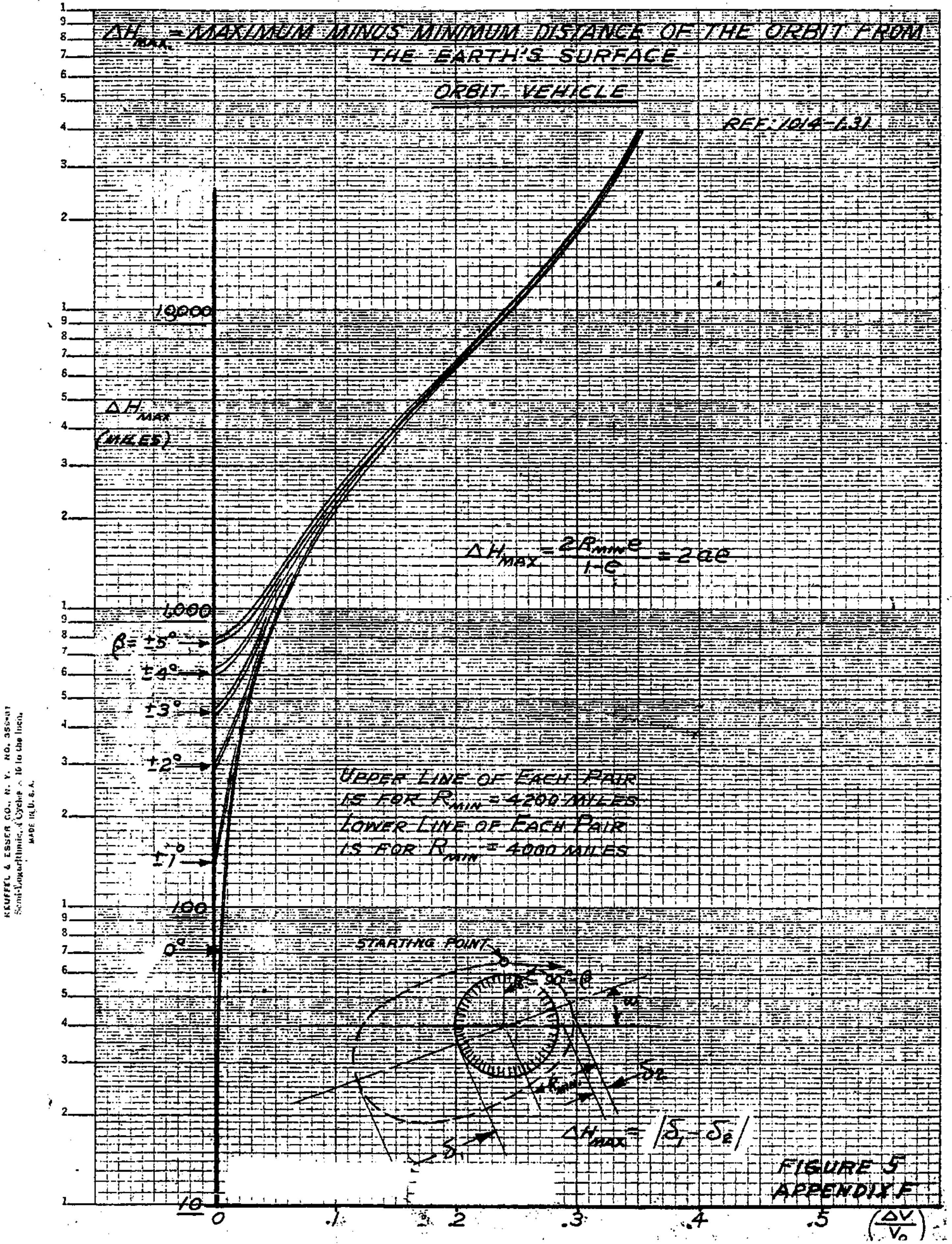


FIGURE 5
APPENDIX F

Analysis ORBIT VEHICLE

Prepared by C.V. STURDEVANT

Date 5-1-46

SANTA MONICA Plant

Page 11 F

Model #1033

Report No. SM 11827

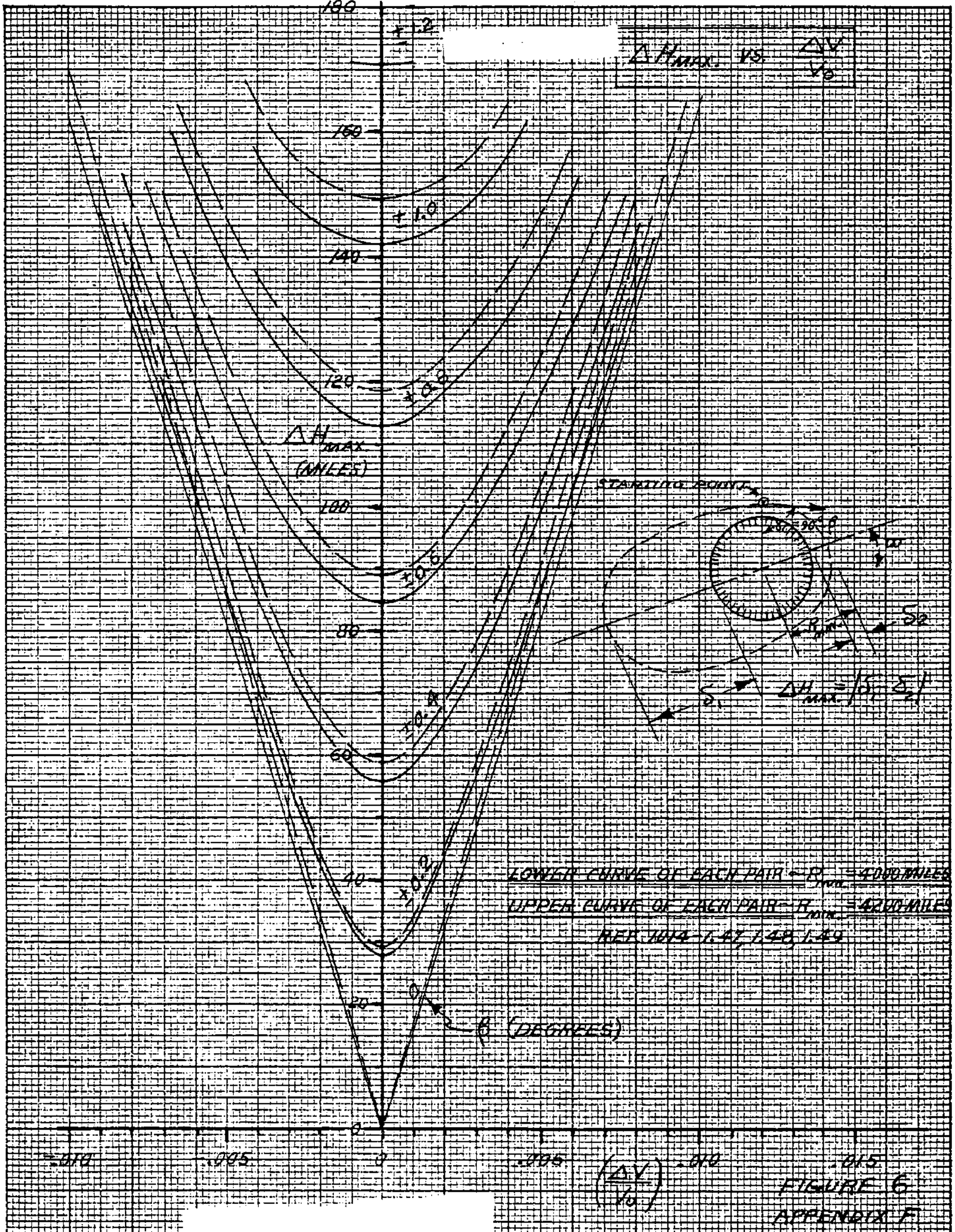


FIGURE 6

APPENDIX F

Appendix F

Development of Approximate Orbital Equations:

Let R = Radius
V = Velocity
β = Inclination to the horizontal
V_c = Equilibrium Velocity in circular orbit at R_o
$\Delta V = V_o - V_c$
$\Delta h = R_o - R_m$
$\delta = \frac{\Delta h}{R_o}$

Subscript zero denotes initial condition

Conservation of energy.

$$\frac{V^2 - V_o^2}{2g_o R_o} = \frac{R_o}{R} - 1$$

Conservation of Angular Momentum

$$\frac{R}{R_o} \times \frac{V}{V_o} \frac{\cos\beta}{\cos\beta_o} = 1$$

Eliminate V

$$\frac{\left(\frac{R_o}{R}\right)^2 V_o^2 \frac{\cos^2\beta_o}{\cos^2\beta} - V_o^2}{2g_o R_o} = \frac{R_o}{R} - 1$$

R is a mininum (or maximum) when $\beta = 0$

$$\frac{\left(\frac{R_o}{R_m}\right)^2 V_o^2 \cos^2\beta_o - V_o^2}{2\, g_o \cdot R_o} = \frac{R_o}{R_m} - 1$$

$$\left(\frac{R_o}{R_m}\right)^2 - \left(\frac{R_o}{R_m}\right)^2 \sin^2\beta_o - 1 = \frac{2\, g_o R_o}{V_o^2}\left(\frac{R_o}{R_m} - 1\right)$$

Appendix F

$$\sin^2\beta_o = 1 - \left(\frac{R_m}{R_o}\right)^2 - \frac{2g_oR_o}{V_o^{\,2}} \times \frac{R_m}{R_o}\left(1 - \frac{R_m}{R_o}\right)$$

Let $\frac{R_m}{R_0} = 1 - \delta$, $V_o = V_c + \Delta V$, β_o small

$$\sin\beta_o = \sqrt{1 - (1-2\delta + \delta^2) \frac{-2g_o R_o \times (1-\delta)\delta}{(V_c^2 + 2V_c\Delta V + \Delta V^2)}}$$

$$\beta_o \cong \sqrt{\delta^2 + 4\delta\frac{\Delta V}{V_c}}, \qquad \beta_o \cong \sqrt{\left(\frac{\Delta h}{R_o}\right)\left[\frac{\Delta h}{R_o} + 4\frac{\Delta V}{V_c}\right]}$$

Appendix F

EQUATIONS FOR CORRECTION OF ORBIT

Let T = Thrust
m = Mass
V = Velocity
I = Impulse
W_f = Fuel Weight
W = Gross Weight
C = Exhaust velocity
β = Inclination to the horizontal

(A) Correction of Angle by Thrust ⊥ to Flight Path

$$T\,dt = m dV$$

$$I = \text{impulse} \cong \beta\, mV$$

$$\Delta W_f = \frac{gI}{C} = \frac{mV\beta g}{C} = \frac{WV\beta}{C}$$

$$\frac{\Delta W_f}{W} = \left(\frac{V}{C}\right)\beta$$

(B) Correction of Velocity by Thrust ∥ to Flight Path

$$T\,dt = m dV$$

$$I = \text{Impulse} = m\Delta V$$

$$\Delta W_f = \frac{gI}{C} = \frac{g}{C} m\Delta V$$

$$\left(\frac{\Delta W_f}{W}\right) = \left(\frac{\Delta V}{C}\right)$$

Appendix F

(C) Correction of Velocity and Angle by Thrust in One Direction

$$T dt \sin\theta = mV$$

$$T dt \cos\theta = m\Delta V$$

$$\left(\frac{mV\beta}{I}\right)^2 + \left(\frac{m\Delta V}{I}\right)^2 = 1$$

$$I = \sqrt{m^2 V^2 \beta^2 + m^2 \Delta V^2} = \frac{\Delta W_f \; C}{g}$$

$$\frac{\Delta W_f}{W} = \sqrt{\left(\frac{V}{C}\right)^2 \beta^2 + \left(\frac{\Delta V}{C}\right)^2}$$

(D) Correction of Angle by Aerodynamic Forces

$$L dt = m dV \qquad L = D \times \left(\frac{L}{D}\right)$$

$$\left(\frac{L}{D}\right) \times D dt = m dV$$

Setting $D = T$

$$\left(\frac{L}{D}\right) \times T dt = m dV$$

$$\frac{\Delta W_f}{W} = \left(\frac{V}{C}\right) \times \beta \times \left(\frac{D}{L}\right)$$

Appendix G

G. THE METEORITE-HIT PROBABILITY FORMULAS

In the section of Chapter 11 dealing with the probability of a meteorite hitting a satellite vehicle, certain probabilities and probability-based time intervals were presented in the Tables. The derivation of the formulas used to compute these quantities is given below.

The meteorites entering the atmosphere will be assumed to have a random distribution both as regards their surface distribution over the atmospheric layer surrounding the earth and as regards their occurrence with time. It is assumed that the meteorites travel through the atmosphere along the vertical and that the planform area of the vehicle is normal to the vertical.

It is not difficult to see that the meteorite velocity and the vehicle velocity are not involved in the computation of the probability that a meteorite will strike the vehicle. This follows essentially from the assumption that the distribution of the meteorites is random with respect to surface area and time. Thus there will be a certain number N of meteorites of specified size which enter the atmosphere in each 24 hour period, and for any exposed area A_b, it is equally likely that this area will be hit regardless of where it may be situated on the surface of the atmospheric shell. This means that the area is equally likely to be hit whether it is moving or stationary and therefore the speed of the moving area is immaterial.

Let the unit of time be the hour and let an event be said to occur when a meteorite hits the vehicle. Then the average number of events $\bar{n}$ which occur in a unit of time (1 hour) is given by

$$\bar{n} = \frac{NA_b}{24A_o}, \qquad (1)$$

Appendix G

and the average time $\bar{t}$ required for the event to occur is the reciprocal of this, or

$$\bar{t} = \frac{1}{\bar{n}} = \frac{24A_e}{NA_b} \text{ . - (2)}$$

It is obvious that the probability of the occurrence of an event must increase as the time t increases. The way in which the time must enter is determined by the Poisson exponential.(1),(2) When an event happens on the average once in the time $\bar{t}$, the average number $\bar{m}$ of events in the time T is

$$\bar{m} = \frac{T}{\bar{t}} = \frac{NA_b}{24A_e} T \text{ , - (3)}$$

Then, according to the Poisson distribution, the probability p_r that the event will happen exactly r times in the time interval T is given by

$$p_r = \frac{\bar{m}^r e^{-\bar{m}}}{r!} \text{ , - (4)}$$

where e is the exponential, e = 2.71828. Further, the probability p_1 that the event will happen exactly once in the time T is

$$p_1 = \bar{m}\, e^{-\bar{m}} \text{ . - (5)}$$

The probability p_o that the event will fail to happen in the time T (i.e. for the event to happen zero times) is, from (4),

$$p_o = p(0) = e^{-\bar{m}} \text{ . - (6)}$$

(1) Kenney, J. F.: Mathematics of Statistics. D. Van Nostrand Co., New York, 1941, p. 29 Part 2.

(2) Freeman, H. A.: Industrial Statistics. John Wiley and Sons, New York, 1942, p. 149.

Appendix G

If p_{1+} denote the probability that the event occur at least once in the time T it follows that $p_{1+} + p_o = 1$ and therefore

$$p_{1+} = 1 - e^{-\bar{m}}. \qquad (7)$$

Inserting the value for $\bar{m}$, the value for this probability becomes

$$p_{1+} = 1 - [e]^{-\frac{NA_b}{24A_e}T} \qquad (8)$$

This value of p_{1+} gives the probability that the vehicle will be hit at least once in T hours. It does not exclude the possibility that more than one hit will occur in this time interval, and in fact, definitely allows that more than one hit may occur. However, although p_{1+} does not specify the probability of the exact number of hits in the time T it is considered to best represent the type of probability which is most significant since, from Eq. (8), it is seen that $p_{1+} = 0$ for N or T = 0 and that p_{1+} increases as N and T increase.

The probability p_1, on the other hand, which from (5) may be written

$$p_1 = \frac{NA_b}{24A_e} T [e]^{-\frac{NA_b}{24A_e}T}, \qquad (9)$$

does exclude the possibility of more than one hit and refers only to exactly one hit, no more and no less. It is evident the p_1 is a much more restricted type of probability than p_{1+} and has the odd characteristic of becoming smaller when the number N or T is very large. This of course follows from the fact that since p_1 refers only to exactly one hit, when N or T become larger and larger and there are thus more and more chances

Appendix G

for a hit, the chances that the vehicle will be hit only once will become smaller.

The third probability of interest p_o, the probability for no hit at all, is evaluated from

$$p_o = [e]^{-\frac{NA_b}{24A_e}T} \quad \text{- - - - - - - - - - - - - - - (10)}$$

Comparing Eqs. (8), (9), and (10), it is seen that as the time T increases, p_{1+} approaches 1, p_1 must go through a maximum, and p_o approaches 0.

Certain probability-based time intervals of interest are as follows.

(a) The time such that the vehicle has a 50 to 50 chance of not being hit. In this case p_o = 0.5 and the corresponding time to satisfy this condition will be denoted by T (0.5). From (10), this is evaluated from

$$T\ (0.5) = -\frac{24A_e}{NA_b}\log_e 0.5 = -\frac{9.96 \times 10^{13}}{N}, \quad \text{- - - - - - - (11)}$$

where the value $\frac{24A_e}{A_b} = 1.437 \times 10^{14}$ is used.

(b) The time such that the vehicle has a 100 to 1 chance of not being hit. In this case p_o = 0.99, and denoting this time by T (0.99) it follows from (10) that

$$T\ (0.99) = -\frac{24A_e}{NA_b}\log_e 0.99 = -\frac{1.437 \times 10^{12}}{N}. \quad \text{- - - - - - - (12)}$$

(c) The time such that the vehicle has a 1,000 to 1 chance of not being hit. In this p_o = 0.999 and the corresponding time T (0.999) is given by

$$T\ (0.999) = -\frac{24A_e}{NA_b}\log_e 0.999 = -\frac{1.437 \times 10^{11}}{N}. \quad \text{- - - - - (13)}$$

Appendix G

Before presenting the various probabilities, a few remarks will be made concerning the number N. In computing the probabilities of a hit, one may consider either the total number of meteorites of all sizes, or only the total number of a certain size, or else the total number of a certain size plus all those of larger size. In considering the probabilities of a hit by a meteorite we are not especially concerned with the entire total number of meteorites of all possible sizes since many of these are too small to do any damage. On the other hand we are concerned with meteorites of a certain given size and especially those sizes which can cause damage and at the same time occur with considerable frequency. Furthermore, since when considering a certain given size the total number of all larger sizes may be appreciable, this suggests also the consideration of the total number of meteorites down to and including those of given size.

The number of meteorites for these two cases are given in Table 1, where meteors of magnitude less than -3 have not been included since they occur too infrequently to be of any importance. The table is based on the fact that when the magnitude increases by 5, the number increases by 10^2 and therefore for a change of 1 magnitude the number changes by a factor of 2.5. Thus, if there are N' meteors of magnitude M' in each 24 hour period, there will be

$$N = N' \times (10)^{\frac{2}{5}(M - M')} \qquad (14)$$

meteors of magnitude M in the same period.

Appendix G

The sum of all those of magnitude -3 up to and including magnitude M is given by

$$S_M = N' \times (10)^{-\frac{2}{5}M'} \sum_{-3}^{M} (10)^{\frac{2}{5}M} \quad \text{------------(15)}$$

Using the relation for the sum of a geometric series this reduces to the form

$$S_M = \frac{10^{\frac{2}{5}}}{10^{\frac{2}{5}} - 1} \times N' \times (10)^{\frac{2}{5}(M - M')} \left[1 - (10)^{-\frac{2}{5}(M + 4)} \right] \quad \text{-----(16)}$$

Choosing a meteor of magnitude 0 as a basis for the computation, $M' = 0$, $N' = 450,000$, and the numbers are then computed from the relation

$$N = 4.5 \times 10^5 \times (10)^{\frac{2M}{5}}, \text{ and} \quad \text{------------------(17)}$$

$$S_M = 7.47 \times 10^5 \left[(10)^{\frac{2M}{5}} - .025 \right] \quad \text{---------------- (18)}$$

Comparing the values of N and S_M in Table 1, it is seen that for magnitude 10 for example the number S_M is about 66 per cent greater than N, a not inconsequential increase.

Appendix G

TABLE 1

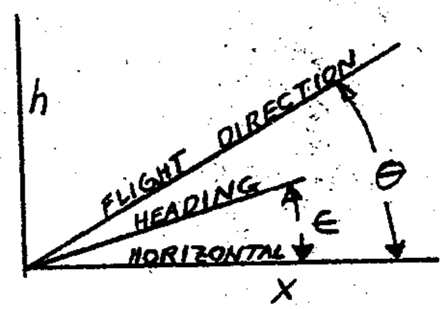
PARTIAL AND TOTAL NUMBER OF METEORITES

Magnitude M	Number of Magnitude M (N)	Total Number from -3 Up to and Including Magnitude M (S_M)
-3	2.84×10^4	2.84×10^4
0	4.5×10^5	7.28×10^5
2	2.84×10^6	4.72×10^6
5	4.5×10^7	7.47×10^7
6	1.132×10^8	1.88×10^8
7	2.84×10^8	4.72×10^8
8	7.14×10^8	1.18×10^9
9	1.795×10^9	2.98×10^9
10	4.5×10^9	7.47×10^9
12	2.84×10^{10}	4.72×10^{10}
15	4.5×10^{11}	7.47×10^{11}
20	4.5×10^{13}	7.47×10^{13}
25	4.5×10^{15}	7.47×10^{15}
30	4.5×10^{17}	7.47×10^{17}

$N = 4.5 \times 10^5 \times 10^{\frac{2M}{5}}$, from Eq. (17).

$S_M = 7.47 \times 10^5 \left[10^{\frac{2M}{5}} - .025 \right]$, from Eq. (18).

Appendix H



H. Development of Stability Equations

(A) Final Stage

T = thrust

m = mass

V = velocity (average)

R = radius from center of earth

M = pitching moment

I = Moment of inertia

d = displacement of thrust axis from C.G.

$$M = I\ddot{\epsilon}$$

$$T\sin\epsilon + \frac{mV^2}{R} - W = m\ddot{h} \doteq mV\dot{\theta}$$

$$\epsilon \doteq \frac{mV\dot{\theta}}{T} + \frac{W}{T} - \frac{mV^2}{RT}$$

$$\text{LET } \frac{M}{I} = K_0\int\left[\theta + \Delta t_0\dot{\theta} + \frac{(\Delta t_0)^2}{2!}\ddot{\theta} + \right]dt + K_1\left[\theta + \Delta t_1\dot{\theta} + \frac{(\Delta t_1)^2}{2!}\ddot{\theta} + \right]$$

$$+ K_2\left[\dot{\theta} + \Delta t_2\ddot{\theta} + \frac{(\Delta t_2)^2}{2!}\dddot{\theta} + \right] + K_3\left[\epsilon + \Delta t_3\dot{\epsilon} + \frac{(\Delta t_3)^2}{2!}\ddot{\epsilon} + \right]$$

$$+ K_4\left[\dot{\epsilon} + \Delta t_4\ddot{\epsilon} + \frac{(\Delta t_4)^2}{2!}\dddot{\epsilon} + \right] + \frac{Td}{I} = \ddot{\epsilon}$$

Where K_1, K_2 etc represent magnitudes of artificially applied restoring and damping moments and K_0 is an integral term necessary for approaching the correct flight path angle when eccentric thrust is present. The terms Δt_0, Δt_1 etc correspond to time lags in application of these moments.

Differentiating the equation and eliminating ϵ

$$K_0\left[\theta+\Delta t_0\dot{\theta}+\frac{\Delta t_0^2}{2!}\ddot{\theta}+\right]+K_1\left[\dot{\theta}+\Delta t_1\ddot{\theta}+\frac{\Delta t_1^2}{2!}\dddot{\theta}+\right]$$

$$+K_2\left[\ddot{\theta}+\Delta t_2\dddot{\theta}+\frac{\Delta t_2^2}{2!}\ddddot{\theta}+\right]+K_3\frac{mV}{T}\left[\ddot{\theta}+\Delta t_3\dddot{\theta}+\frac{\Delta t_3^2}{2!}\ddddot{\theta}+\right]$$

$$+K_4\frac{mV}{T}\left[\dddot{\theta}+\Delta t_4\ddddot{\theta}+\frac{\Delta t_4^2}{2!}\overset{\cdot\cdot\cdot\cdot\cdot}{\theta}+\right]-\frac{mV}{T}\ddddot{\theta}=0$$

For small time lags the equation becomes:

$$\ddddot{\theta}+\dddot{\theta}\left[-K_4\right]+\ddot{\theta}\left[-K_3-\frac{K_2T}{mV}\right]+\dot{\theta}\left[-\frac{K_1T}{mV}\right]+\theta\left[-\frac{K_0T}{mV}\right]=0$$

The conditions for stability are that all coefficients of θ, $\dot{\theta}$ etc must be positive and that

$$-\frac{K_1T}{mV}\left\{\left[-K_3-\frac{K_2T}{mV}\right]\left[-K_4\right]+\frac{K_1T}{mV}\right\}-\left[-K_4\right]^2\left[-\frac{K_0T}{mV}\right]>0$$

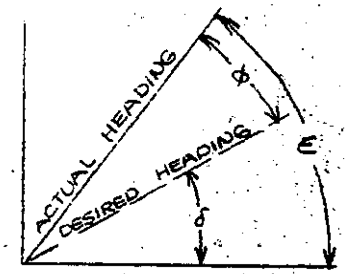
Either K_3 or K_2 can be omitted without causing instability. To simplify analysis omit K_2 which would probably be the more difficult term to apply in practice.

Since all values of K are normally negative it is evident that the first stability condition is satisfied. Rearranging the second condition and assuming negative values for K_0, K_1, etc gives:

$$\left|K_3\right|>\left|\frac{K_1}{K_4}\frac{T}{mV}\right|+\left|\frac{K_4}{K_1}K_0\right|$$

This indicates that K_3 should be large, and (if a value of K_0 is established) that $\frac{K_1}{K_4}$ should approach $\sqrt{\frac{mV}{T}K_0}$

(B) Initial Stages



M = pitching moment

I = moment of inertia

T = thrust

d = displacement of thrust axis from C.G.

ϵ = actual heading

δ = design heading as a predetermined function of time

$$M = I\ddot{\epsilon}$$

Neglecting time lag let

$$\frac{M}{I} = K_0\int(\epsilon-\delta)dt + K_1(\epsilon-\delta) + K_2(\dot{\epsilon}-\dot{\delta}) + \ddot{\delta} + \frac{Td}{I} = \ddot{\epsilon}$$

Where K_1 and K_2 represent magnitudes of artificially applied restoring and damping moments, and K_0 is an integral term necessary for approaching the exact desired heading when eccentric thrust is present. The term $\ddot{\delta}$ is artificially applied as a predetermined function of time.

Differentiating the equation

$$K_0(\epsilon-\delta) + K_1(\dot{\epsilon}-\dot{\delta}) + K_2(\ddot{\epsilon}-\ddot{\delta}) - (\dddot{\epsilon}-\dddot{\delta}) = 0$$

$$\text{OR} \quad \dddot{\phi} + \ddot{\phi}(-K_2) + \dot{\phi}(-K_1) + \phi(-K_0) = 0$$

where $\phi = \epsilon - \delta$

For stability the coefficients of ϕ, $\dot{\phi}$ etc must all be positive, and since K_0, K_1 and K_2 are normally negative this condition is satisfied

Also for stability

$$(-K_2)(-K_1) = -(-K_0) > 0$$

$$\text{OR} \quad |K_2 K_1| > |K_0|$$

(C) General

Future investigations of stability might well be approached in a somewhat different manner. Since these systems are stabilized by entirely artificial means it would seem desirable to stipulate the type of motion desired and the damping and then determine the necessary values of K_1 K_2 etc to accomplish this. This approach should be mathematically simpler also.